Vector

Any directed line segment is called vector. Here direction is from A to B. A : Initial point - IP
B : Final / Terminal point - TP
|AB| : magnitude of AB

Magnitude: Distance between initial point A and Terminal point.

Direction: From IP → TP

Sense (Line of action): The straight line along which vector acts.

Types of vectors: The vector having magnitude '1' is called unit vector.

1) Unit vector: The vector having magnitude '1' is called unit vector $\frac{\vec{AB}}{|\vec{AB}|}$. $\hat{AB}$ direction of unit vector is along given vector.

2) Null vector: The vector which is of 0 magnitude ... It has arbitrary direction hence, it's direction is undefined.
$\vec{0} + \vec{A} = \vec{A}$
$\vec{A} + \vec{0} = \vec{A}$

3) Parallel vectors: Two vectors are parallel, if their lines of action are parallel.

Collinear: When two vectors are parallel they are said to be collinear. Parallel vectors are said to be collinear. Two vectors A & B are parallel if a = mb ...

Co-INITIAL: Two or more vectors are said to be co-initial if they lie in same plane have same initial point.
e.g. are coinitial vectors
$\vec{AB}, \vec{AC}, \vec{AD}$

Angle between two vectors is measured when they are co-initial.

Co-planar: 3 or more vectors are said to be co-planar if they lie in same plane.

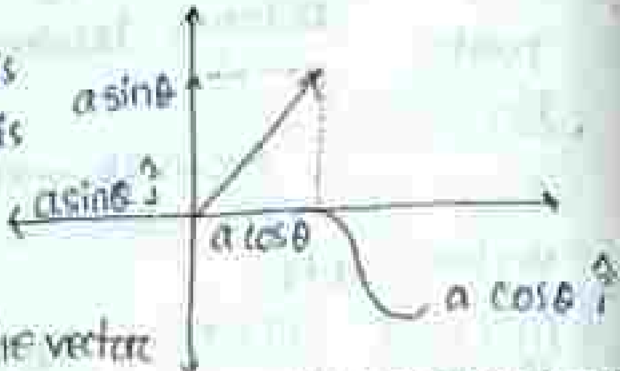
$\vec{a}\ \vec{b}\ \vec{c}$ are coplanar volume of solid generated by them
$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

If three vectors $\vec{a}, \vec{b}\ \vec{c}$ are coplanar if $\lambda\vec{a} + \mu\vec{b} = \vec{c}$

Localised / free vector :- The vector whose initial point is fixed is called localised vector.

RESOLUTION OF A VECTOR:

$\hat{i}$ = unit vector along +ve X axis
$\hat{j}$ = " " " +ve Y axis



$\vec{a} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j}$

Q.1 Find X and Y component of the vector whose magnitude is 10 and which makes angle 60° at +ve X-axis?

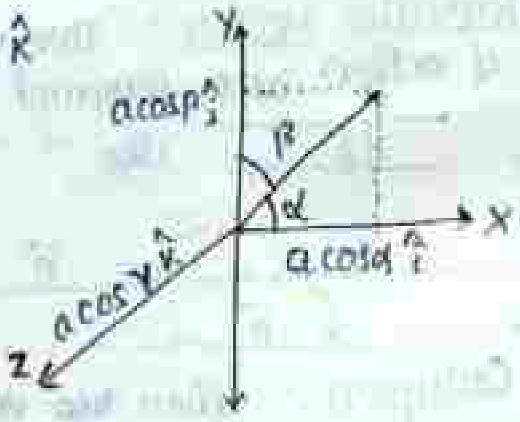
$|\vec{a}| = 10$
$\theta = 60°$

X - component $= 10\cos\theta\, 60\hat{i}$
$= 10 \times \frac{1}{2}\hat{i} = 5\hat{i}$

Y - component $= 10\sin\theta\, 60\hat{j}$
$= 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3}\,\hat{j}$

$\therefore \vec{a} = 5\hat{i} + 5\sqrt{3}\hat{j}$

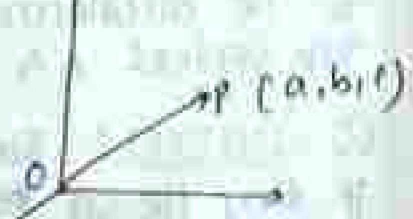
→ $\vec{a} = a\cos\alpha\,\hat{i} + a\cos\beta\,\hat{j} + a\cos\gamma\,\hat{k}$



POSITION VECTOR:-

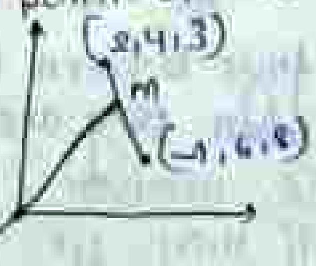
position vector of a point 'P' is the vector from origin that point ($\vec{OP}$)

position vector of $P = \vec{OP}$
$= a\hat{i} + b\hat{j} + c\hat{k}$



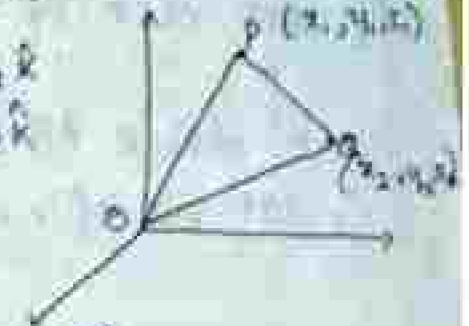
Q.2 Find the position vector of the mid point of the line joining (2,4,3), (-1,6,8)

co-ordinate of $M = \left(\frac{-1+2}{2}, \frac{6+4}{2}, \frac{8+3}{2}\right)$
$= \left(\frac{1}{2}, 5, \frac{11}{2}\right)$



position vector of $M = \vec{OM}$
$= \frac{1}{2}\hat{i} + 5\hat{j} + \frac{11}{2}\hat{k}$ (Ans)

→ position vector of $P = \vec{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$
$Q = \vec{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$



In Δ OPQ $\vec{OP} + \vec{PQ} = \vec{OQ}$
$\Rightarrow \vec{PQ} = \vec{OQ} - \vec{OP}$
$= pos(Q) - pos(P)$
$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Q.3 Find $\vec{PQ}$ and its scalar & vector components where $P = (3,-4,6)$ and $Q = (8,7,11)$

$\vec{PQ} = (8-3)\hat{i} + [7-(-4)]\hat{j} + (11-6)\hat{k}$
$= 5\hat{i} + 11\hat{j} + 6\hat{k}$

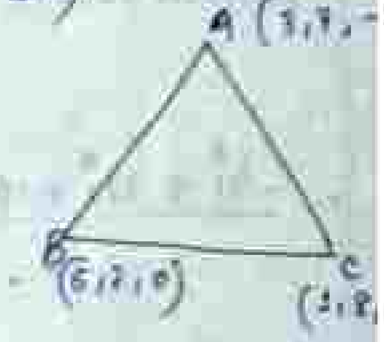
$\therefore$ Scalar components are $(5, 11, 6)$
vector " " $(5\hat{i}, 11\hat{j} + 5\hat{k})$

Q.4
$\vec{AB} = (5-3)\hat{i} + (2-7)\hat{j} + [0-(-1)]\hat{k}$
$= 2\hat{i} - 5\hat{j} + \hat{k}$

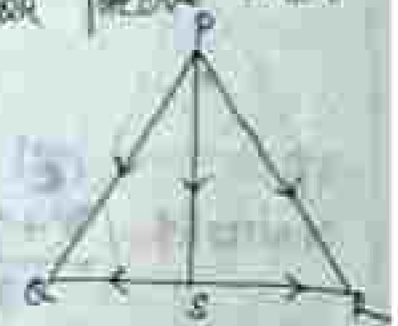
$\vec{BC} = (1-5)\hat{i} + (8-2)\hat{j} + (3-0)\hat{k}$
$= -4\hat{i} + 6\hat{j} + 3\hat{k}$

$\vec{AC} = (1-3)\hat{i} + (8-7)\hat{j} + [3-(-1)]\hat{k}$
$= -2\hat{i} + \hat{j} + 4\hat{k}$

$\vec{CA} = (3-1)\hat{i} + (7-8)\hat{j} + (-1-3)\hat{k}$
$= 2\hat{i} - \hat{j} - 4\hat{k}$



Q.5 In Δ PQR, S is the mid point of QR prove that $\vec{PQ} + \vec{PR} = 2\vec{PS}$



Sol^n
$\vec{QS} = \vec{SR}$
$\vec{SQ} + \vec{SR} = 0$

In Δ PQS $\vec{PS} + \vec{QS} = \vec{PQ}$
In Δ PRS $\vec{PS} + \vec{SR} = \vec{PR}$

Adding the equation
$\vec{PS} + \vec{QS} = \vec{PQ}$
$+\ \vec{PS} + \vec{SR} = \vec{PR}$
$2\vec{PS} + (\vec{QS} + \vec{SR}) = \vec{PQ} + \vec{PR}$
$\Rightarrow 2\vec{PS} + 0 = \vec{PQ} + \vec{PR} \Rightarrow \vec{PQ} + \vec{PR} = 2\vec{PS}$ (proved)

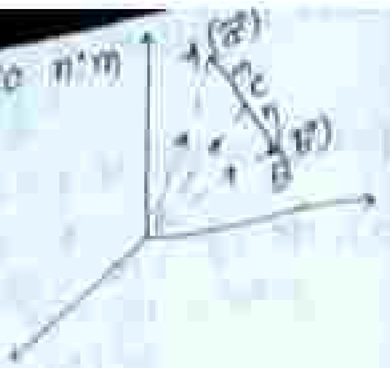
Note: c divides AB in the ratio n : m

pos vect of A : $\vec{a}$

pos vect of B : $\vec{b}$

c : divide AB in m : n

pos vec of c :



Find the vector from origin to the point the line joining (3, 4, 1) & (−2, 6, 8) in the ratio 2 : 5 :

Position vector of A = $\vec{a}$ = $3\hat{i} + 4\hat{j} + \hat{k}$

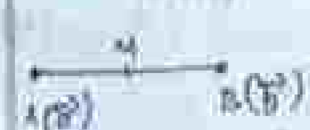
Position vector of B = $\vec{b}$ = $-2\hat{i} + 6\hat{j} + 8\hat{k}$

'c' divides AB in 2 : 5

position vector of c = $\vec{OC} = \frac{2\vec{b} + 5\vec{a}}{2+5}$

$= \frac{2(-2\hat{i} + 6\hat{j} + 8\hat{k}) + 5(3\hat{i} + 4\hat{j} + \hat{k})}{7}$

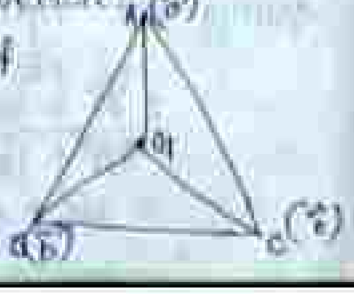
$= \frac{-4\hat{i} + 12\hat{j} + 16\hat{k} + 15\hat{i} + 20\hat{j} + 5\hat{k}}{7} = \frac{11\hat{i} + 32\hat{j} + 21\hat{k}}{7}$



mid point $= \frac{\vec{a} + \vec{b}}{2}$

centroid $\left(\frac{\vec{a} + \vec{b} + \vec{c}}{3}\right)$

Prove that the sum of the vectors from centroid to the vertices of a triangle is a null vector.



Let ABC be a Δ with position vector of A, B, C as $\vec{a}, \vec{b}, \vec{c}$ respective

Let 'G' be the centroid of Δ

Claim: $\vec{GA} + \vec{GB} + \vec{GC} = \vec{0}$

As 'G' is the centroid position vector of G = $\frac{\vec{a} + \vec{b} + \vec{c}}{3}$

$\vec{GA}$ = position vector of A − position vector of G

$= \vec{a} - \frac{\vec{a} + \vec{b} + \vec{c}}{3}$

$\vec{GB}$ = position vector of B − position vector of G

$= \vec{b} - \frac{\vec{a} + \vec{b} + \vec{c}}{3}$ ~~$= \frac{2\vec{b} - \vec{a} - \vec{c}}{3}$~~

$\vec{GC}$ = position vector of C − position vector of G

$= \vec{c} - \frac{\vec{a} + \vec{b} + \vec{c}}{3}$ ~~$= \frac{2\vec{c} - \vec{a} - \vec{b}}{3}$~~

$\vec{GA} + \vec{GB} + \vec{GC}$

$= \vec{a} - \frac{\vec{a} + \vec{b} + \vec{c}}{3} + \vec{b} - \frac{\vec{a} + \vec{b} + \vec{c}}{3} + \vec{c} - \frac{\vec{a} + \vec{b} + \vec{c}}{3}$

$= \vec{a} + \vec{b} + \vec{c} - 3\left(\frac{\vec{a} + \vec{b} + \vec{c}}{3}\right)$

$= \vec{0}$

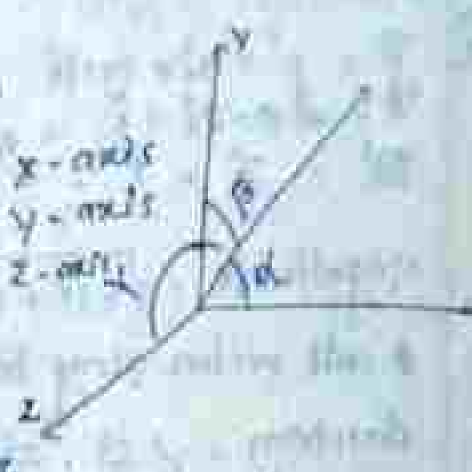
Direction of a vector :-

$\alpha, \beta, \gamma \rightarrow$ direction angles

α − angle between vector and x-axis

β − " " " y-axis

γ − " " " z-axis



$\langle \cos\alpha, \cos\beta, \cos\gamma \rangle$

direction cosines (dcs) of vector

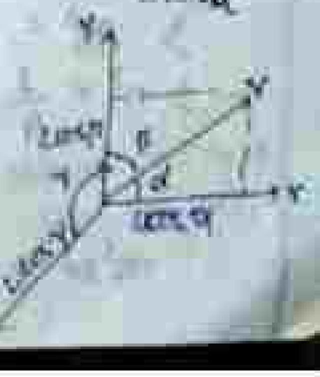
In 2D: $(l\cos\alpha)^2 + (l\cos\beta)^2 = l^2$

$\Rightarrow l^2\cos^2\alpha + l^2\cos^2\beta = l^2$

$\Rightarrow \cos^2\alpha + \cos^2\beta = 1$

In 3D: $(l\cos\alpha)^2 + (l\cos\beta)^2 + (l\cos\gamma)^2 = l^2$

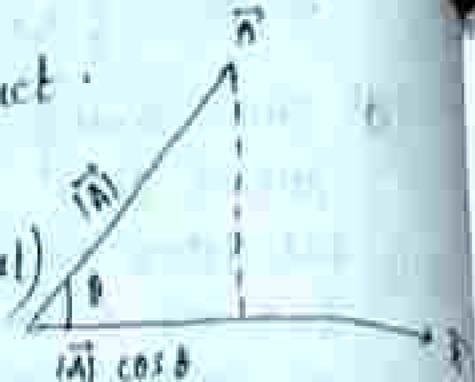
$\boxed{\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$



Dot Product :-

- It is called SCALAR product.

dot product = (magnitude of first). (projection of sec on first)



$\Rightarrow \vec{A}\cdot\vec{B} = |\vec{B}|\cdot|\vec{A}|\cos\theta$

(1) $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

(2) $\vec{A}\cdot\vec{B} = 0$ $\left(\Rightarrow \vec{A} = \vec{0} \text{ or } \vec{B} = \vec{0} \text{ or } \theta = 90^\circ\right)$

(3) $\hat{i}\cdot\hat{j} = 0$, $\hat{j}\cdot\hat{k} = 0$, $\hat{i}\cdot\hat{k} = 0$

(4) $|\vec{A}|\cdot|\vec{A}| = |\vec{A}||\vec{A}|\cos 0 = |\vec{A}|^2$

(5) $\hat{i}\cdot\hat{i} = |\hat{i}|^2 = 1^2 = 1$

$\hat{j}\cdot\hat{j} = 1$, $\hat{k}\cdot\hat{k} = 1$

(6) $\vec{A}, \vec{B}$, how to find $\vec{A}\cdot\vec{B}$

$|\vec{A}| = m$

$|\vec{B}| = n$ $\boxed{\vec{A}\cdot\vec{B} = mn\cos\theta}$

angle betn $\vec{A}$ & $\vec{B}$ $= \theta$

$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\boxed{\vec{A}\cdot\vec{B} = a_1b_1 + a_2b_2 + a_3b_3}$

$\Rightarrow \vec{A}\cdot\vec{B} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$= a_1b_1 + a_2b_2 + a_3b_3$

Q-1 $\vec{A} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$\vec{B} = 4\hat{i} + 7\hat{k}$

$\vec{A}\cdot\vec{B} = 3\cdot4 + (-2)(0) + (6)(7)$

$= 12 + 42 = 54$

Q-2 If $|\vec{a}| = 3$, $|\vec{b}| = 5$, find max & min value of $\vec{a}\cdot\vec{b}$

$\Rightarrow \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = 3\times5\cos\theta = 15\cos\theta$

$\vec{a}\cdot\vec{b}]_{max} = 15$ $(\because \cos\theta = 1)$

$\vec{a}\cdot\vec{b}]_{min} = -15$ $(\because \cos\theta = -1)$

Q-3 For what value of λ, the vector $2\hat{i} - 3\hat{j} + \lambda\hat{k}$ & $4\hat{i} + 7\hat{j} + 2\hat{k}$ are perpendicular?

If the vectors are $\perp$, dot product = 0

$\Rightarrow 8 + (-21) + 2\lambda = 0$

$\Rightarrow 8 - 21 + 2\lambda = 0 \Rightarrow -13 + 2\lambda \Rightarrow \lambda = \frac{13}{2} = 6.5$

Q-4 A body under a force $\vec{F} = 3\hat{i} - 5\hat{j} - 2\hat{k}$ moves from A $(-2, 3, 1)$ to B $(6, 1, 9)$. Find the work done!

$\vec{s} = \vec{AB} = \vec{B} - \vec{A}$

$= (6+2)\hat{i} + (1-3)\hat{j} + (9-1)\hat{k}$

$= 8\hat{i} - 2\hat{j} + 8\hat{k}$

$W = \vec{F}\cdot\vec{s}$

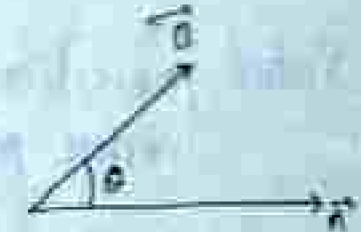
$= (3\times8) + (-5)(-2) + (-2)8$

$= 24 + 10 - 16 = 18$ unit

(7) Maximum value of $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|$

Minimum value of $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$

Angle between two Vectors :-



$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta$

$\boxed{\Rightarrow \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}}$

$\Rightarrow \cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$

Q-5 Find the angle between the vector :

(i) $3\hat{i} + 4\hat{j} - \hat{k}$, $2\hat{i} + 2\hat{j} + \hat{k}$

(ii) $\hat{i} + \hat{j}$, $\hat{j} - \hat{k}$

Let $\vec{a} = 3\hat{i} + 4\hat{j} - \hat{k}$

$\vec{b} = 2\hat{i} + 2\hat{j} + \hat{k}$

Let 'θ' be the angle between $\vec{a}$ & $\vec{b}$

we know that $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$= \dfrac{(3\times1)+(4\times2)+(-1)(2)}{\sqrt{3^2+4^2+(-1)^2}\,\sqrt{1^2+2^2+2^2}}$

$= \dfrac{3+8-1}{\sqrt{26}\;\sqrt{9}} = \dfrac{13}{3\sqrt{26}}$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{13}{3\sqrt{26}}\right)$

(4) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{(1\times0)+(1\times1)+(0\times1)}{\sqrt{1^2+(1)^2+(0)^2}\,\sqrt{(0)^2+(1)^2+(1)^2}}$

$\Rightarrow \cos\theta = \dfrac{1}{2}$ $\Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{2}\right)$

$\Rightarrow \theta = 60^\circ$

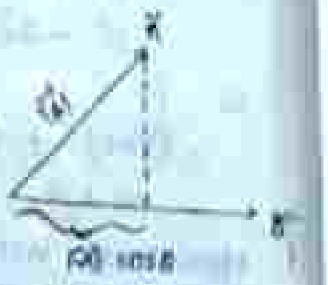
Projection :-

Scalar projection :-

projection of $\vec{A}$ on $\vec{B} = |\vec{A}|\cos\theta$

$= \dfrac{|\vec{B}|\,|\vec{A}|\cos\theta}{|\vec{B}|}$

$= \dfrac{\vec{A}\cdot\vec{B}}{|\vec{B}|} = \vec{A}\cdot\hat{B}$



Vector projection :-

vector projection of $\vec{A}$ on $\vec{B}$

$= \left(\dfrac{\vec{A}\cdot\vec{B}}{|\vec{B}|}\right)\cdot\hat{B} = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{B}|}\cdot\dfrac{\vec{B}}{|\vec{B}|}$

$= \dfrac{(\vec{A}\cdot\vec{B})\,\vec{B}}{|\vec{B}|^2}$

Q. Find scalar & vector projection of $\vec{a} = 6\hat{i}+3\hat{j}+\hat{k}$ on $\vec{b} = 3\hat{i}+2\hat{j}-6\hat{k}$

scalar projection of $\vec{a}$ on $\vec{b}$ = $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \dfrac{(6\times3)+(3\times2)+(1\times-6)}{\sqrt{3^2+2^2+(-6)^2}}$

$= \dfrac{18+6-6}{\sqrt{49}} = \dfrac{18}{7}$

vector projection of $\vec{a}$ on $\vec{b}$ = $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\cdot\hat{b}$

$= \dfrac{18}{7}\cdot\dfrac{3\hat{i}+2\hat{j}-6\hat{k}}{\sqrt{(3)^2+(2)^2+(-6)^2}}$

$= \dfrac{18}{7}\cdot\dfrac{3\hat{i}+2\hat{j}-6\hat{k}}{7}$

$= \dfrac{18}{49}(3\hat{i}+2\hat{j}-6\hat{k})$

Cross product $(\vec{A}\times\vec{B})$

It is denoted by $\vec{A}\times\vec{B}$

It is called vector product

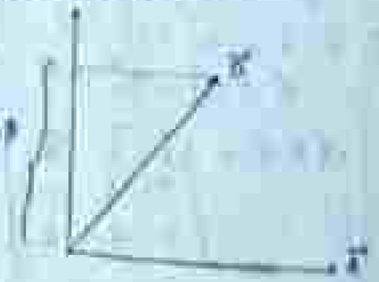
Magnitude of $\vec{A}\times\vec{B}$ is defined as magnitude of 1 vector × projection of second vector perpendicular to 1st vector

$|\vec{A}\times\vec{B}| = |A|\,|B|\sin\theta$

Direction of $\vec{A}\times\vec{B}$ is perpendicular to both $\vec{A}$ & $\vec{B}$

(According to clockwise or anticlock wise)

$\vec{A}\times\vec{B} = |\vec{A}|\,|\vec{B}|\sin\theta\,\hat{n}$



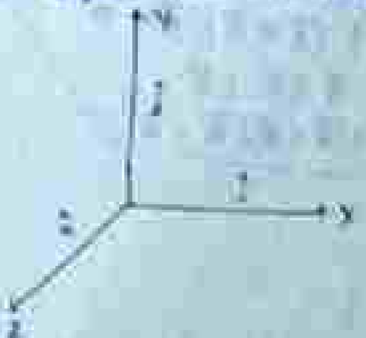
$\vec{A}\times\vec{B}$ maximum value is $|\vec{A}|\,|\vec{B}|$

$\vec{A}\times\vec{B} = \vec{0}$ when

(i) $\vec{A}\parallel\vec{B}$ ($\theta = 0^\circ$, $\theta = 180^\circ$)

(ii) If atleast 1 of $\vec{A}$, $\vec{B}$ is a zero/null vector

$\hat{i}\times\hat{j} = \hat{k}$
$\hat{j}\times\hat{k} = \hat{i}$
$\hat{k}\times\hat{i} = \hat{j}$



* $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$ (direction are opposite)

$|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}|$

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ $[\vec{A}\times\vec{A} = 0]$

Evaluation of cross product :

$\vec{a} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

$\vec{b} = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$\vec{a}\times\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$= \vec{0} + a_1b_2\hat{k} - a_1b_3\hat{j} - a_2b_1\hat{k} + \vec{0} + a_2b_3\hat{i} + a_3b_1\hat{j} - a_3b_2\hat{i} + \vec{0}$

$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

If $\vec{a} = 3\hat{i} + 5\hat{j} - 6\hat{k}$

$\vec{b} = 4\hat{i} - 7\hat{j} + 2\hat{k}$

$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & -6 \\ 4 & -7 & 2 \end{vmatrix} = \hat{i}\begin{vmatrix} 5 & -6 \\ -7 & 2 \end{vmatrix} \hat{j}\begin{vmatrix} 3 & -6 \\ 4 & 2 \end{vmatrix} \hat{k}\begin{vmatrix} 3 & 5 \\ 4 & -7 \end{vmatrix}$

$= \hat{i}(10 - 42) - \hat{j}(6 + 24) + \hat{k}(-21 - 20)$

$= -32\hat{i} - 30\hat{j} - 41\hat{k}$

$\vec{a}\times\vec{b} \perp \vec{a}$

$\Rightarrow (\vec{a}\times\vec{b})\cdot\vec{a} = 0$

$\vec{a}\times\vec{b} \perp \vec{b}$

$\Rightarrow (\vec{a}\times\vec{b})\cdot\vec{b} = 0$

$[\vec{a}\cdot(\vec{b}\times\vec{c})]^2$

$= [\vec{a}\cdot|\vec{b}||\vec{c}|\sin 90^\circ \cdot \hat{a}]^2$

$= [\vec{a}\cdot bc \cdot 1\, \hat{a}]^2$

$= b^2c^2[\vec{a}\cdot\hat{a}]^2$

$= b^2c^2[|\vec{a}||\hat{a}|\cos 0]^2$

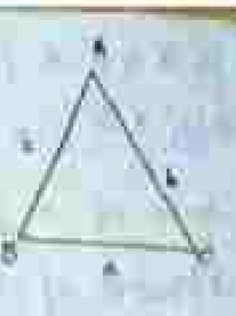
$= b^2c^2[a\cdot 1\cdot 1]^2$

$= a^2b^2c^2$

Area of Δ

Area of $\Delta = \frac{1}{2}|\vec{AB}\times\vec{BC}| = \frac{1}{2}|\vec{BC}\times\vec{AC}|$

$= \frac{1}{2}|\vec{AC}\times\vec{AB}|$



Q. Find area of the triangle whose vertices are (1, 2, 3)A, (3, 4, 6)B, (2, 7, 8)C

Sol:
$\vec{BA} = 2\hat{i} - 2\hat{j} - 3\hat{k}$

$\vec{BC} = 3\hat{i} + 3\hat{j} + 2\hat{k}$

$\vec{BA}\times\vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & -3 \\ 3 & 3 & 2 \end{vmatrix}$

$= \hat{i}\begin{vmatrix} -2 & -3 \\ 3 & 2 \end{vmatrix} + \hat{j}\begin{vmatrix} -2 & -3 \\ 3 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -2 \\ 3 & 3 \end{vmatrix}$

$= 5\hat{i} - 13\hat{j} + 12\hat{k}$

$|\vec{AB}\times\vec{BC}| = \sqrt{(5)^2 + (13)^2 + (12)^2}$

$= \sqrt{(13)^2 + (13)^2} = 2\sqrt{13}$ $\qquad \frac{1}{2}\sqrt{b^2a^2 + b^2a^2 + c^2a^2}$

Area of $\Delta = \frac{1}{2}\times 2\sqrt{13} = \sqrt{13}$

Area of Parallelogram

(i) Area of parallelogram

$= 2\times\frac{1}{2}|\vec{a}\times\vec{b}|$

$= |\vec{a}\times\vec{b}|$

(ii) ⇒ formula $= \frac{1}{2}|\vec{d_1}\times\vec{d_2}|$

proof: Area $= |\vec{a}\times\vec{b}|$

$\vec{d_1} = \vec{a} + \vec{b}$

$\vec{d_2} = \vec{a} - \vec{b}$

$\vec{d_1}\times\vec{d_2} = (\vec{a} + \vec{b})\cdot(\vec{a} - \vec{b})$

$\vec{d_1}\times\vec{d_2} = \vec{0} - \vec{a}\times\vec{b} + \vec{b}\times\vec{a} - \vec{0}$

$\vec{d_1}\times\vec{d_2} = -\vec{a}\times\vec{b} - \vec{a}\times\vec{b}$

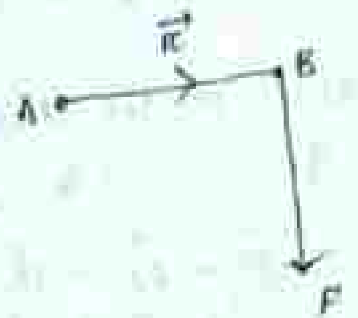
$|\vec{a_1} \times \vec{a_2}| = |-2\,\vec{a} \times \vec{b}|$

$\Rightarrow \frac{|\vec{a_1} \times \vec{a_2}|}{2} = \vec{a} \times \vec{b}$

Area of $= \frac{1}{2} |\vec{a_1} \times \vec{a_2}|$

Moment of a force :-

Moment of $F = \vec{r} \times \vec{F}$ at 0



F is applied at B

moment of F at A $= \vec{AB} \times \vec{F}$

Scalar Triple product (STP)

- Triple product is a scalar
- $\vec{a}, \vec{b}, \vec{c}$ scalar triple product is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$
- $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$

volume of parallelopiped whose edges are $\vec{a}, b, c$ is

$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

- $\cdot$ & $\times$ canbe interchange in $\vec{a}\ \vec{b}\ \vec{c}$ scalar triple product
- $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

E-

$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$[\vec{A}\ \vec{B}\ \vec{c}] = \vec{A} \cdot (\vec{B} \times \vec{c})$

$\vec{B} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\hat{i} - (b_1c_3 - b_3c_1)\hat{j} + \hat{k}(b_1c_2 - b_2c_1)$

$\vec{A} \cdot (\vec{B} \times \vec{c}) = a_1(b_2c_3 - b_3c_2)$

Q- Find $[\vec{A}\ \vec{B}\ \vec{c}]$ where $\vec{a} = 2\hat{i} + \hat{j}$, $\vec{b} = 2\hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{k}$

$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$

$= 2(4-0) - 1(0-1)$

$= 8 + 1 = 9$ (cubic)

Q- Find volume of parallelopiped whose 3 coterminous edg

$\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$

$c = 3\hat{i}$

$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & 3 & 6 \\ 1 & 2 & -1 \\ 3 & 0 & 0 \end{vmatrix}$

$= -3\begin{vmatrix} 2 & 6 \\ 1 & -1 \end{vmatrix} = -3(-2-6)$

$= -3(-8)$

$= 24$ cubic unit

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

(i) $\vec{a}, \vec{b}, \vec{c}$ are coplanar

(ii) At least one of the vectors $\vec{a}, \vec{b}, \vec{c}$ is a null vector

$[\vec{0}\ \vec{b}\ \vec{c}] = 0$, $[\vec{a}\ \vec{0}\ \vec{a}] = 0$

(iii) If any two are parallel (equal)

Q- Find value of λ if the vectors, $\vec{a} = 3\hat{i} - 2\hat{j} + \lambda\hat{k}$, $\vec{b} = 4\hat{i} + \hat{j} + \hat{k}$, $\vec{c} = 2\hat{i} + 2\hat{j} + \hat{k}$

If $\vec{a}, \vec{b}, \vec{c}$ are coplanar, $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$\Rightarrow \begin{vmatrix} 3 & -2 & \lambda \\ 4 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 0$ $\Rightarrow 3(-1) + 2(2) + \lambda(6) = 0$

$\Rightarrow -3 + 4 + 6\lambda = 0$

$\Rightarrow +1 + 6\lambda \Rightarrow \lambda = \frac{-1}{6}$

Q. prove that scalar triple product of $[\vec{a}+\vec{b} \;\; \vec{b}+\vec{c} \;\; \vec{c}+\vec{a}] = 2[\vec{a}\;\vec{b}\;\vec{c}]$

Sol. $[\vec{a}+\vec{b} \;\; \vec{b}+\vec{c} \;\; \vec{c}+\vec{a}]$

$= (\vec{a}+\vec{b}) \cdot \{(\vec{b}+\vec{c}) \times (\vec{c}+\vec{a})\}$

$= (\vec{a}+\vec{b}) \cdot \{\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{c} + \vec{c}\times\vec{a}\}$ $[\because \vec{c}\times\vec{c} = 0]$

$= (\vec{a}+\vec{b}) \cdot \{\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a}\}$

$= \vec{a}\cdot(\vec{b}\times\vec{c}) + \vec{a}\cdot(\vec{b}\times\vec{a}) + \vec{a}\cdot(\vec{b}\times\vec{a}) + \vec{b}\cdot(\vec{b}\times\vec{c}) +$
$\vec{b}\cdot(\vec{b}\times\vec{a}) + \vec{b}\cdot(\vec{c}\times\vec{a})$ $[\because \vec{p}\times\vec{q} \perp \vec{p}, \vec{q}]$

$= \vec{a}\cdot(\vec{b}\times\vec{c}) + 0+0+0+0 + \vec{b}\cdot(\vec{c}\times\vec{a})$

$= [\vec{a}\;\vec{b}\;\vec{c}] + [\vec{b}\;\vec{c}\;\vec{a}] = [\vec{a}\;\vec{b}\;\vec{c}] + [\vec{a}\;\vec{b}\;\vec{c}]$

$= 2[\vec{a}\;\vec{b}\;\vec{c}]$ (proved)

Q. prove that the following vector are coplanar ! when the vectors are $\hat{i} - 2\hat{j} + 2\hat{k}$, $3\hat{i} + 4\hat{j} + 5\hat{k}$, $-2\hat{i} + 4\hat{j} - 4\hat{k}$

$\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

$\vec{b} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

$\vec{c} = -2\hat{i} + 4\hat{j} - 4\hat{k}$

$\Rightarrow \begin{vmatrix} 1 & -2 & 2 \\ 3 & 4 & 5 \\ -2 & 4 & -4 \end{vmatrix} = 0$

$\Rightarrow 1[4\times -4 - 20] - (-2)[-12 - (-10)] +$
$2[12 - (-8)] = 0$

$\Rightarrow [-16 - 20] + 2[-10 + 12] + 2[-20] = 0$

$\Rightarrow -36 + 4(1) + 40 = 0$

$\Rightarrow -40 + 40 = 0$ (proved)

$[\vec{a}\;\vec{b}\;\vec{c}]$ are coplanar.

Continuity.

A function $f(x)$ is said to be continuous at $x = c$, if

LHL at $x=c$ = RHL at $x=c$ = function value at $x = c$

i.e., $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$

But we know that,

when $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = \lambda$

$\Rightarrow \lim_{x\to c} f(x) = \lambda$

So, continuity at $x=c$ may be defined as $\lim_{x\to c} f(x) = f(c)$

Q. check continuity of $f(x) = \begin{cases} \frac{x^3 - 8}{x-2}, & x \neq 2 \\ 12, & x = 2 \end{cases}$ at $x = 2$.

LHL of $f(x)$ at $x=2$

$\lim_{x\to 2^-} f(x)$

$= \lim_{h\to 0} f(2-h)$

$= \lim_{h\to 0} \frac{(2-h)^3 - 8}{2-h-2}$

$= \lim_{h\to 0} \frac{2^3 - 3\cdot 2^2 h + 3\cdot 2h^2 - h^3 - 8}{-h}$

$= \lim_{h\to 0} \frac{-12h + 6h^2 - h^3}{-h}$

$= \lim_{h\to 0} \frac{-h(12 - 6h + h^2)}{-h}$ $= 12 - 6\times 0 + 0^2$
$= 12$

LIMIT & CONTINUITY

Notation

$\forall$ → For all

$\exists$ → There Exist

$\exists!$ → There Exist Uniquely

$\in$ → Belongs to

$\subset$ → Super Subset

$\subseteq$ → Subset

S.t. → Such that

$f(x)$ → function of x

$f: X \to Y$ → A mapping or function from X to Y

D_f → Domain of function

R/R_f → Range of function

$\mathbb{R}$ → Set of real numbers.

$\mathbb{Z}$ → Set of integers

$\mathbb{N}$ → Set of natural numbers

$\mathbb{Q}$ → Set of rational numbers

$\mathbb{Q}^c$ → Set of irrational numbers

Function

A function is a special case of relation.
A function f from X to Y i.e. $f: X \to Y$ s.t. for each $x \in X$ $\exists!$ $y \in Y$ which is related to x by the relation f.
We write $y = f(x)$ & call it image of x under f & call x is the preimage of y under f.

Onto or Surjective f^n

A function $f: X \to Y$ is said to be an onto or surjective f^n if $R_f = f(X) = Y$ i.e. every element of Y is the image of some element of X.

Into function

A function $f: X \to Y$ is said to be an into function if $R_f = f(X) \subset Y$ i.e. if $\exists$ at least one element $y \in Y$ which has no preimage in X.

One-One or injective f^n

A function $f: X \to Y$ is said to be 1-1 or one-one or injective function if for every $x_1, x_2 \in X$

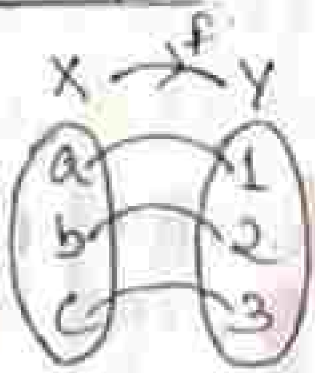
$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2$

Many one f^n

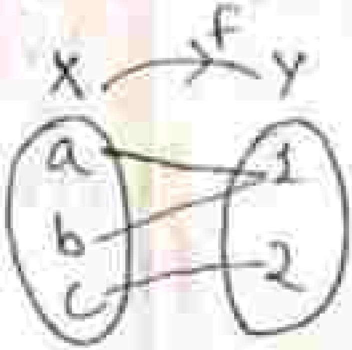
A function $f : X \to Y$ is said to be many-one f^n if $\exists\, x_1, x_2 \in X$ with $x_1 \neq x_2$ s.t. $f(x_1) = f(x_2)$

Examples

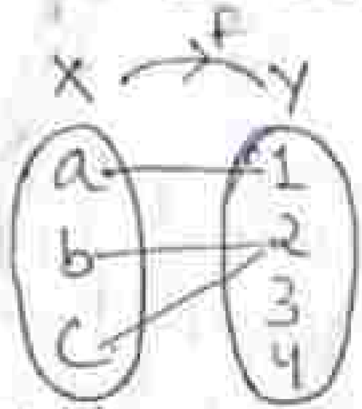


(i) one-one & onto

(ii) One-one & into



(iii) Many-one & onto



(iv) Many-one & into

Domain

The set X is known as Domain of the function.

Co-Domain

The set Y is called the co-domain of the function.

Range

The set of all images of the elements of x under the mapping or function f is called the range of f & denoted by $f(x)$.

Real valued function

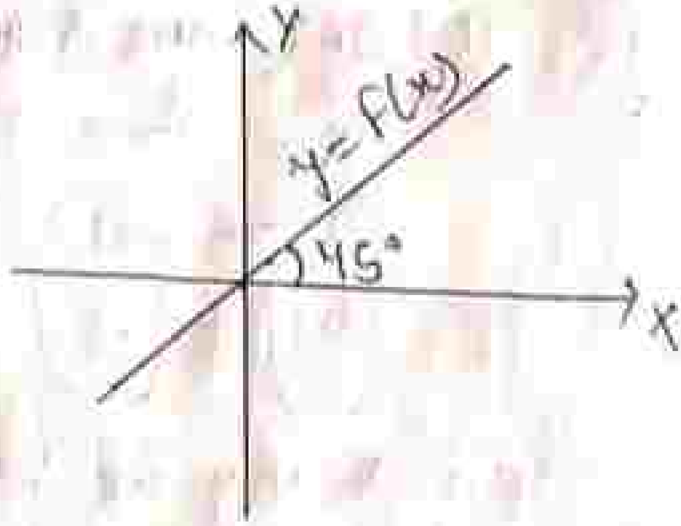
A function $f: X \rightarrow Y$ is said to be real valued f^n if $X, Y \in \mathbb{R}$.

TYPES OF FUNCTION

(i) **Identity function**

The function f is defined by $f(x) = x$ $\forall x \in R$ is called the identity function.

Here, $D_f = R_f = \mathbb{R}$



(ii) **Constant function**

The function f defined $f(x) = c$ $\forall x \in D_f$ where c is any real constant number.

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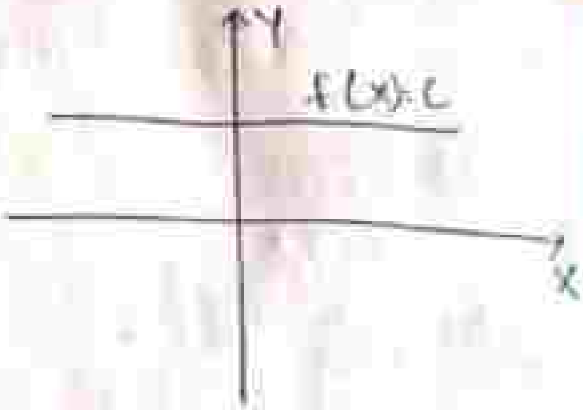


Modulus function

The function f defined by $f(x) = |x|$

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$ is called the modulus function or absolute value function $D_f = \mathbb{R}$, $R_f = [0, \infty)$

Greatest integer fuction

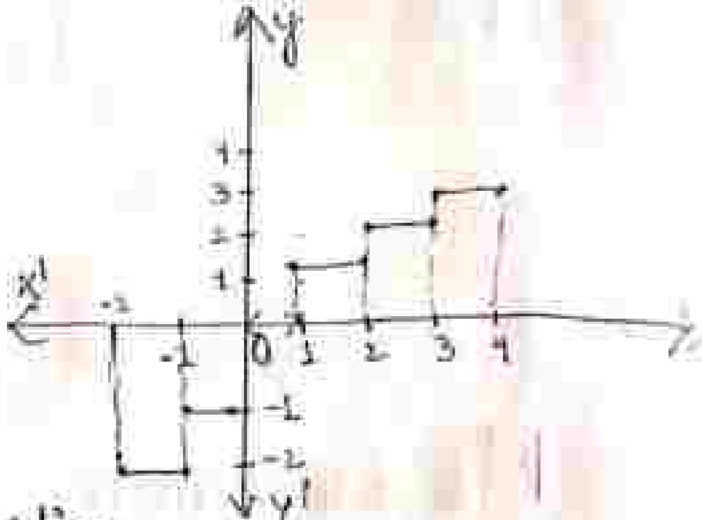
For every $x \in \mathbb{R}$, $[x]$ is greatest integer function $\leq x$.

Let n be an integer. Then

$$[x] = \begin{cases} n, & \text{if } x = n \\ n-1, & \text{if } n-1 \leq x \leq n \end{cases}$$

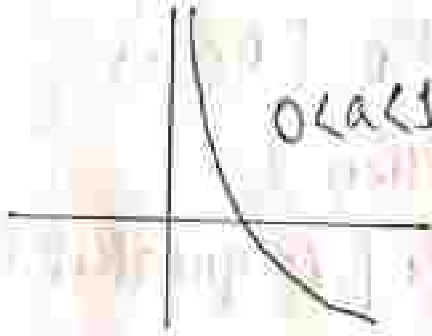
Eg $[3] = 3$, $[-4] = -4$, $[\frac{19}{3}] = 6$, $[\sqrt{2}] = 1$

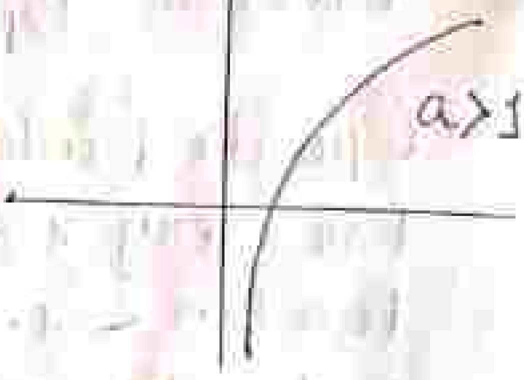
x	...	$-2 \leq x < -1$	$-1 \leq x < 0$	$0 \leq x < 1$	$1 \leq x < 2$	$2 \leq x < 3$
y	...	-2	-1	0	1	2



Logarithmic Function

The function f defined by $f(x) = \log_a x$, where $x > 0$, $a > 0 \neq 1$, $a \in \mathbb{R}$ is called logarithmic function.

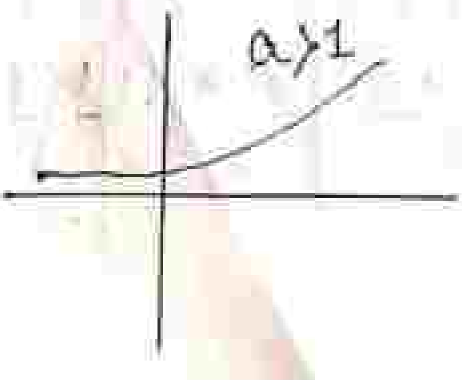




Exponential Function

The function f defined by $f(x) = a^x$ where $a > 0$ & $x \in \mathbb{R}$ is called exponential function

$D_f = \mathbb{R}$, $R_f = (0, \infty)$



Trigonometric Function

Function	Domain	Range
$\sin x$	$\mathbb{R}$	$[-1, 1]$
$\cos x$	$\mathbb{R}$	$[-1, 1]$
$\tan x$	$\mathbb{R} - \{(2n+1)\pi/2, n \in Z\}$	$\mathbb{R}$
$\cot x$	$\mathbb{R} - \{n\pi, n \in Z\}$	$\mathbb{R}$
$\sec x$	$\mathbb{R} - \{(2n+1)\pi/2, n \in Z\}$	$\mathbb{R} - (-1, 1)$
$\operatorname{cosec} x$	$\mathbb{R} \{n\pi/2, n \in Z\}$	$\mathbb{R} - (-1, 1)$

Expansion of Formula

① $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

② $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$

③ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$

④ $a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \cdots$

⑤ $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

⑥ $\sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot x^5 + \cdots$

Some Important Limit

① $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

Proof: $\lim_{x \to a} \left(\frac{x^n - a^n}{x - a}\right) = \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1})}{(x-a)}$

$= \lim_{x \to a} (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1})$

$= a^{n-1} + a^{n-2}.a + a^{n-3}a^2 + \cdots + a^{n-1}$

$= a^{n-1} + a^{n-1} + a^{n-1} + \cdots + a^{n-1}$

$= na^{n-1}$

② $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

Proof: $\lim_{x \to 0} \frac{\log(1+x)}{x}$

Putting $\log(1+x) = y$

$\Rightarrow 1 + x = e^y$

$\Rightarrow x = e^y - 1$

When, $x \to 0$, $y \to 0$.

$\lim_{y \to 0} \frac{y}{e^y - 1}$

$$= \lim_{y \to 0} \frac{1}{\frac{e^y - 1}{y}} = 1 \quad \left(\because \lim_{y \to 0} \frac{e^y - 1}{y} = 1 \right)$$

③ $\lim_{x \to 0} (1+x)^{1/x} = e$

Proof. $LHS = \lim_{x \to 0} (1+x)^{1/x}$

$$= \lim_{x \to 0} \left(1 + \frac{1}{x} \cdot x + \frac{1/x(1/x - 1)}{2!} x^2 + \frac{\frac{1}{x}(1/x - 1)(1/x - 2)}{3!} x^3 + \cdots \right)$$

$$= \lim_{x \to 0} \left(1 + 1 + \frac{1-x}{2!} + \frac{2(1-x)(1-2x)}{3!} + \cdots \right)$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e^1$$

④ $\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$

Proof Let us assume $a^x - 1 = y$

$\Rightarrow a^x = y + 1$

$\Rightarrow \log a^x = \log (y+1)$

$\Rightarrow x \log a = \log (y+1)$

$\Rightarrow x = \frac{\log (y+1)}{\log a}$

as, $x \to 0$, $y \to 0$

$$\lim_{y \to 0} \frac{y}{\frac{\log(y+1)}{\log a}}$$

$$= \lim_{y \to 0} \frac{y \log a}{\log(y+1)}$$

$$\Rightarrow \lim_{y \to 0} \frac{\log a}{\frac{\log(y+1)}{y}}$$

$$= \lim_{y \to 0} \frac{\log a}{\frac{\log(y+1)}{y}}$$

$$= \frac{\log a}{1} \qquad \left[\because \lim_{y \to 0} \frac{\log(1+y)}{y} = 1 \right]$$

(5) $$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$= \lim_{x \to 0} \left[\frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) - 1}{x} \right]$$

$$= \lim_{x \to 0} \frac{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)}{x}$$

$$= \lim_{x \to 0} \frac{x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots \right)}{x}$$

$$= 1 + 0 + 0 + 0 + \cdots$$

$$= 1$$

(6) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Proof. $\lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x}$

$= \lim_{x \to 0} \frac{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)}{x}$

$= \lim_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)$

$= 1 - 0 + 0 - \cdots$

$= 1$

(7) $\lim_{x \to 0} \cos x = 1$

Proof. We know that $\cos x = 1 - 2\sin^2\frac{x}{2}$

$\therefore \lim_{x \to 0} \cos x = \lim_{x \to 0} \left[1 - 2\sin^2\frac{x}{2}\right]$

$= 1 - 2\left[\lim_{x \to 0}\left(\sin\frac{x}{2}\right)^2\right]$

$= 1 - 2(0)^2 \qquad \left[\because \lim_{x \to 0} \sin\frac{x}{2} = 0\right]$

$= 1$

⑦ $\lim_{x \to 0} \frac{\tan x}{x} = 1$

Proof

$$\lim_{x \to 0} \frac{\tan x}{x}$$

$$= \lim_{x \to 0} \left[\frac{\sin x}{x} \cdot \frac{1}{\cos x} \right]$$

$$= \left(\lim_{x \to 0} \frac{\sin x}{x} \right) \cdot \lim_{x \to 0} \frac{1}{\cos x}$$

$$= (1 \times 1)$$

$$= 1$$

CONTINUITY

→ A function $f(x)$ is said to be continuous at $x = a$ if

(i) $\lim_{x \to a} f(x)$ exist

(ii) $f(a)$ exist

(iii) $\lim_{x \to a} f(x) = f(x)$

Eg. $f(x) = \begin{cases} \frac{x - 4x + 3}{x - 1}, & x \neq 1 \\ -2, & x = 1 \end{cases}$

at $x = 1$ test continuity

Sol

$\lim_{x \to 1^-} \frac{x^2 - 4x + 3}{x - 1}$

$= \lim_{x \to 1^-} \frac{x^2 - 3x - x + 3}{x - 1}$

$= \lim_{x \to 1^-} \frac{x(x-3) - 1(x-3)}{(x-1)}$

$= \lim_{x \to 1^-} \frac{(x-3)(x-1)}{(x-1)}$

$= \lim_{h \to 0} (1 - h - 3) = -2$

RHL

$$\lim_{x\to 1^+} \frac{x^2-4x+3}{x-1}$$

$$= \lim_{x\to 1^+} \frac{x^2-3x-x+3}{x-1}$$

$$= \lim_{x\to 1^+} \frac{x(x-3)-1(x-3)}{x-1}$$

$$= \lim_{x\to 1^+} \frac{(x-3)(x-1)}{(x-1)}$$

$$= \lim_{h\to 0} (1+h-3) = -2$$

$$\lim_{x\to 1} f(x) = -2 = f(a)$$

∴, Hence the f^n. $f(x)$ is continuous at $x=1$.

Q. $f(x) = \begin{cases} 2x+1 & \text{if } x \le 0 \\ x & \text{if } 0 < x \le 1 \\ 2x-1 & \text{if } x > 1 \end{cases}$ at $x=0$

Sol^n: Here $a = 0$

LHL

$$\lim_{x\to 0^-} 2x+1 = \lim_{h\to 0} 2(0-h)+1$$

$$= 1$$

RHL

$$\lim_{x\to 0^+} x = \lim_{h\to 0} 0+h$$

$$= 0$$

∴, Hence LHL ≠ RHL

So, the given f^n is no continuous at $x = 0$

at $x = 1$

Soln. Here, $a = 1$

LHL

$$\lim_{x \to 1^-} x = \lim_{h \to 0} 1 - h$$

$$= 1$$

RHL

$$\lim_{x \to 1^+} 2x - 1 = \lim_{h \to 0} 2(1+h) - 1$$

$$= 1$$

$$\lim_{x \to 1} f(x) = 1 = f(a)$$

∴, Hence LHL = RHL

So, the given f^n is continuous at $x = 1$

Q. $f(x) = \begin{cases} -\cos x, & x \geq 0 \\ \cos x, & x < 0 \end{cases}$ at $x = 0$ test continuity

Q. $f(x) = \begin{cases} \frac{x}{|x|} & \text{when } x \neq 0 \\ 1 & x = 0 \end{cases}$ at $x = 0$ test continuity

DIFFERENTIAL CALCULUS

Differentiation = process

Derivative = Result of the process

Let,

$y = f(x)$

x = variable changing

Changing y to $y + \delta y$ or $f(x + \delta y)$

Now finding the rate of change in 'y' with respect to 'x'

$$\frac{f(x+\delta x) - f(x)}{x + \delta y - x}$$

$$\frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \frac{y + \delta y - y}{\delta x}$$

$$= \frac{\delta y}{\delta x}$$

$$\lim \frac{\delta y}{\delta x} = \frac{f(x+\delta y) - f(x)}{\delta x}$$

$$\lim_{\delta x \to 0} \frac{\delta x}{\delta y}$$

$$\Rightarrow \frac{f(x+\delta x) - f(x)}{\delta x}$$

Now the changing in rate

$$\Rightarrow \frac{\delta y}{\delta x}$$

Now $\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$

Derivative :—

Derivative is the result of the process called as the differentiation of a function.

$$\frac{dy}{dx} = y = f(x)$$

Formulae :—

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} a^x = a^x \ln a$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \cot x = \operatorname{cosec}^2 x$$

$$\frac{d}{dx} \sec x = \sec x \cdot \tan x$$

$$\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x$$

Find the derivatives of the following

1. $y = 2x^3 - 3x^2 + 4x - 5$

2. $y = x^3 + e^x + 3^x + \cot x$

3. $y = 9x^2 + 3/x + 5\sin x$.

4. $y = x^2 + \frac{4}{x^2} - 2/3 \tan x + 7\log_e x + 6e^x$

5. $y = \log_e x$

Answer -

1) $y = 2x^3 - 3x^2 + 4x - 5$

$\frac{dy}{dx} = 6x^2 - 6x + 4$

2) $y = x^3 + e^x + 3^x + \cot x$

$\frac{dy}{dx} = 3x^2 + e^x + 3^x \ln 3 + -\operatorname{cosec}^2 x$

3) $y = 9x^2 + 3/x + 5\sin x$

$\frac{dy}{dx} = 18x - 3/x^2 + 5\cos x$

4) $y = x^2 + 4/x^2 - 2/3 \tan x + 7\log_e x + 6e^x$

$\frac{dy}{dx} = 2x - 8/x^3 - 2/3 \sec^2 x + 7/x + 6e^x$

5) $y = \log_e x$

$dy/dx = \frac{1}{x}$

Find the derivatives :—

1) $y = \dfrac{1 - \tan x}{1 + \tan x}$

2) $y = \sqrt{\dfrac{1 - \cos 2x}{1 + \cos 2x}}$

3) $y = \sqrt{1 + \sin 2x}$

4) $y = x \sin x$ [illegible] $- \dfrac{e^x}{1 + x^2}$

1) $y = \dfrac{1 - \tan x}{1 + \tan x}$

$\dfrac{d}{dx}(y) = \dfrac{d}{dx}\left(\dfrac{1 - \tan x}{1 + \tan x}\right)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{(1 + \tan x)\frac{d}{dx}(1 - \tan x) - (1 - \tan x)\frac{d}{dx}(1 \ldots}{(1 + \tan x)^2}$

$= \dfrac{(1 + \tan x)(-\sec^2 x) - (1 - \tan x)(\sec^2 \ldots}{(1 + \tan x)^2}$

$= \dfrac{-\sec^2 x - \tan x \cdot \sec^2 x - (\sec^2 x - \sec^2 \ldots}{(1 + \tan x)^2}$

$= \dfrac{-\sec^2 x - \tan x \cdot \sec^2 x - \sec^2 x + \sec^2 \ldots}{(1 + \tan x)^2}$

$= \dfrac{-2\sec^2 x}{(1 + \tan x)^2}$

2) $y = \sqrt{\dfrac{1 - \cos 2x}{1 + \cos 2x}}$

$\dfrac{dy}{dx} = \dfrac{d}{dx}\sqrt{\left(\dfrac{1 - \cos 2x}{1 + \cos 2x}\right)}$

$\Rightarrow \dfrac{dy}{dx} = \sqrt{\dfrac{2\sin^2 x}{2\cos^2 x}}$

$= \sqrt{\tan^2 x}$

$\dfrac{dy}{dx} = \tan x$

$dy/dx = \sec^2 x$ (Ans)

3) $y = \sqrt{1 + \sin 2x}$

$= \sqrt{(\sin x + \cos x)^2}$

$= \sin x + \cos x$

$\dfrac{dy}{dx} = \dfrac{d}{dx}(\sin x + \cos x)$

$= \cos x - \sin x$ (Ans)

4) $y = x\sin x - \dfrac{e^x}{1 + x^2}$

$\dfrac{dy}{dx} = x \dfrac{d}{dx}\sin x + \sin x \dfrac{d}{dx} x - \dfrac{\left[(1+x^2)\dfrac{d}{dx}e^x - e^x \dfrac{d}{dx}(1+x^2)\right]}{(1+x^2)^2}$

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$$\frac{dy}{dx} = x\cos x + \sin x - \frac{\left[e^x(1+x^2) - e^x \cdot 2x\right]}{(1+x^2)^2}$$

$$= x\cos x + \sin x - \frac{e^x(1+x^2) - 2xe^x}{(1+x^2)^2}$$

5) (a) $\frac{3x^2}{5}$

Let,

$y = \frac{3x^2}{5}$

$$\frac{dy}{dx} = \frac{5\frac{d}{dx}3x^2 - 3x^2\frac{d}{dx}5}{(5)^2}$$

$$= \frac{5 \times 6x}{25}$$

$$= \frac{30x}{25}$$

2) let,

$y = 3x^3 + 9x + 9$

$\frac{dy}{dx} = 9x^2 + 9$

$$y = \frac{x^4}{4} + 5x + 2/9$$

$$\frac{dy}{dx} = \frac{4 \times 4x^3}{16} + 5$$

$$= \frac{16x^3}{16} + 5$$

$$= x^3 + 5$$

CHAIN RULE :-

If y is a function of $f(x)$ and also $f(x)$ is a function of $g(x)$ then $y = f[g(x)]$ then y and the process to solve this type of question is taken by chain rule.

Example :-

$$y = \sqrt{x^3+4}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x^3+4}\right)$$

$$= (x^3+4)^{1/2}$$

$$= 1/2\,(x^3+4)^{-1/2}\,\frac{d}{dx}(x^3+4)$$

$$= \frac{3x^2}{2\sqrt{x^3+4}}$$

2) $y = \frac{1}{(x^3+4x)^2}$

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{[x^3+4x]^2}\right)$

$= (x^3+4x)^{-2}$

$= -2(x^3+4x)^{-3}\frac{d}{dx}(x^3+4x)$

$= \left[-2(x^3+4x)^{-3}\right](3x^2+4)$

$= (3x^2+4)\left[-2(x^3+4x)^{-3}\right]$

$= \frac{-2(3x+4)}{(x^3+4x)^3}$

3) $y = \log(\sin x)$

$\frac{dy}{dx} = \frac{1}{\sin x}\frac{d}{dx}\sin x$

$= \frac{\cos x}{\sin x}$

$= \cot x$

4) $y = \log\sin\sqrt{x}$

$\frac{dy}{dx} = \frac{1}{\sin\sqrt{x}}\cdot\frac{d}{dx}\sin\sqrt{x}$

$= \frac{1}{\sin\sqrt{x}}\cdot\cos\sqrt{x}\,\frac{d}{dx}\sqrt{x}$

$= \frac{\cot\sqrt{x}}{2\sqrt{x}}$

1) $\frac{d}{dx}(\sin x + 2x^3 - 6x) = \cos x + 6x^2 - 6$

2) $\frac{d}{dx}(2x^4 - x^3/4) = 8x^3 - \frac{4 \times 3x^2}{16} = 8x^3 - \frac{12x^2}{164}$

3) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

4) $\frac{d}{dx}(x^3/2x+3) = \frac{(2x+3) \times 3x^2 - 2x^3}{(2x+3)^2}$

5) $\frac{d}{dt}(t^{5/2} - 5/2t + t^{-9}) = 5/2 t^{3/2} - 5/2 - 9t^{-10}$

6) $\frac{d}{dx}\left(\frac{t^{2/9}}{2t^6}\right) = \frac{2t^6 \times 2/9 t^{-7/9} - t^{2/9} \times 12t^5}{(2t^6)^2}$

7) $\frac{d}{dt}(\sec t \times e^t) = e^t \sec t + e^t \cdot \sec t \cdot \tan t$

8) $\frac{d}{dx}(e^t \sin t) = e^t \sin t \cdot \cos t$

4) $\frac{d}{dx}\left(\frac{x^3}{2x+3}\right)$

$= \frac{(2x+3)\, d/dx\, x^3 - x^3 \frac{d}{dx}(2x+3)}{2x+3}$

$= \frac{3x^2(2x+3) - 2x^3}{2x+3}$

$y = e^{\sin x^3 + 3x}$

$\frac{dy}{dx} = e^{\sin x^3 + 3x} \cdot \frac{d}{dx}(\sin x^3 + 3x)$

$= e^{\sin x^3 + 3x} \cdot 3x^2 \cos x^3 + 3$

1) $y = \left(\frac{1+x}{x^2+1}\right)^3$

10) $y = (\log(\log(\log x)))$

2) $y = \sin^2 x$

3) $y = \log(\sin x)$

4) $y = \sqrt{\sin\sqrt{x}}$

5) $y = e^{\tan\sqrt{x}}$

6) $y = \cos^2\sqrt{x}$

7) $y = \sin\sqrt{x}$

8) $y = \sqrt{\sin x}$

9) $y = x e^{\sin x}$

1) $y = \left(\frac{7x}{x^2+1}\right)^3$

$\frac{dy}{dx} = 3\left(\frac{7x}{x^2+1}\right)^2 \frac{d}{dx}\left(\frac{7x}{x^2+1}\right)$

$= 3\left(\frac{7x}{x^2+1}\right)^2 \left[\frac{7(x^2+1) - 7x \times 2x}{(x^2+1)2}\right]$

$= 3\left(\frac{7x}{x^2+1}\right)^2 \left[\frac{7(x^2+1) - 14x^2}{(x^2+1)^2}\right]$

2) $y = \sin^2 x$

$= (\sin x)^2$

$\frac{dy}{dx} = 2\sin x \cos x$

$= \sin 2x$

3) $y = \log(\sin x)$

$\frac{dy}{dx} = \frac{1}{\sin x} \frac{d}{dx} \sin x$

$= \frac{\cos x}{\sin x}$

$= \cot x$

4) $y = \sqrt{\sin\sqrt{x}}$

$= \frac{1}{2}(\sin\sqrt{x})^{-1/2} \frac{d}{dx} \sin\sqrt{x}$

$= \frac{1}{2\sqrt{\sin\sqrt{x}}} \times \frac{\cos\sqrt{x}}{2\sqrt{x}}$

$= \frac{\cos\sqrt{x}}{4\sqrt{\sin\sqrt{x}}\sqrt{x}}$

5)

$y = e^{\tan^{-1} x}$

$\frac{dy}{dx} = e^{\tan^{-1} x} \cdot \frac{d}{dx} \tan^{-1} x$

$= e^{\tan^{-1} x} \cdot$ [illegible]

6)

$y = \cos^{-1} \sqrt{x}$

$(\cos^{-1} \sqrt{x})$ [illegible]

[illegible] $\frac{d}{dx} \cos^{-1} x$

[illegible]

$e^{\log x}$

7)

$y = \sin^{-1} \sqrt{x}$

$\frac{dy}{dx}$ [illegible]

$y = \log$ [illegible]

8)

$y =$ [illegible]

$\frac{dy}{dx}$ [illegible]

9)

$y = e^{\sin x}$

$\frac{dy}{dx} = e^{\sin x} \cdot \cos x$

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$y = \log(\log(\log x))$

$\frac{dy}{dx} = \frac{1}{\log(\log x)} \cdot \frac{d}{dx} \log(\log x)$

$= \frac{1}{\log(\log x)} \times \frac{1}{\log x} \frac{d}{dx} \log x$

$= \frac{1}{\log(\log x)} \cdot \frac{1}{\log x} \cdot \frac{1}{x}$

Differentiate w.r.t 'x'

1. $\log x \cdot e^{\sin x + x^3}$
2. $\sin[\cos(\tan x)]$
3. $\log(\sin x)^{\cos x}$
4. $e^{2x} \sin 3x$
5. $\frac{e^x \log x}{x^2}$

(1) let,

$y = \log x \cdot e^{\sin x + x^3}$

$\frac{dy}{dx} = \log x \frac{d}{dx} e^{\sin x + x^3} + e^{\sin x + x^3} \frac{d}{dx} \log x$

$= \log x \left\{ e^{\sin x + x^3} (\cos x + 3x^2) \right\} + \frac{e^{\sin x + x^3}}{x}$

$= e^{\sin x + x^3} \left[\left\{ \log x (\cos x + 3x^2) \right\} + \frac{1}{x} \right]$

(3) let

$y = \log(\sin x)^{\cos x}$

$\frac{dy}{dx} = \frac{1}{(\sin x)^{\cos x}} \cdot (\cos x)^{\cos x} \times (-\sin x)$

3) let,

$y = \sin[\cos(\tan x)]$

$\frac{dy}{dx} = \cos[\cos(\tan x)]$

$= \cos[-\sin(\tan x)]$

$= \cos[-\sin(\tan x)] \cdot \sec^2 x$

4) let,

$y = e^{2x} \sin 3x$

$\frac{dy}{dx} = e^{2x} \frac{d}{dx} \sin 3x + \sin 3x \frac{d}{dx} e^{2x}$

$= 3\cos 3x \cdot e^{2x} + 2e^{2x} \cdot \sin 3x$

5) let,

$y = \frac{e^x \log x}{x^2}$

$\frac{dy}{dx} = \frac{x^2 \frac{d}{dx}(e^x \cdot \log x) - e^x \log x \frac{d}{dx} x^2}{x^4}$

$= \frac{x^2 \left(\frac{e^x}{x} \times \log x \cdot e^x\right) - 2x \cdot e^x}{x^4}$

3) let,

$y = \sin[\cos(\tan x)]$

$\frac{dy}{dx} = \cos[\cos(\tan x)] \times [-\sin(\tan x) \cdot \sec^2 x]$

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③ let,

$y = \log(\sin x)^{\cos x}$

$= \cos x \log \sin x$

$\frac{dy}{dx} = \cos x \frac{d}{dx} \log \sin x + \log \sin x \frac{d}{dx} \cos x$

$= \frac{\cos x}{\sin x} \times \cos x + (-\sin x) \log \sin x$

$= \cot x \cdot \cos x + (-\sin x) \log \sin x$

Exercise:-

① let,

$y = x^2 - 7x + 3$

$\frac{dy}{dx} = \frac{d}{dx}(x^2 - 7x + 3)$

$= 2x - 7$

② let,

$y = \frac{1}{x^2}$

$\frac{dy}{dx} = -\frac{1}{x^3}$

③ let,

$y = \sqrt{\cos x}$

$\frac{dy}{dx} = \frac{\sin x}{2\sqrt{\cos x}}$

④ let,

$y = x \sin x$

$\frac{dy}{dx} = x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} x$

$= x \cos x + \sin x$

⑤ let,

$y = ax^2 + bx + c$

$\frac{dy}{dx} = 2ax + b$

⑥ let,

$y = x(x+2)(x+3)$

$= (x^2 + 2x)(x+3)$

$= x^3 + 3x^2 + 2x^2 + 6x$

$= x^3 + 5x^2 + 6x$

$\frac{dy}{dx} = 3x^2 + 10x + 6$

⑦ $y = x^4 - 2x^2 + 7$

$\frac{dy}{dx} = 4x^3 - 4x$

8) $y = \dfrac{(x+2)(x^2+5)}{x}$

$= \dfrac{x^3+5x+2x^2+10}{x}$

$\dfrac{dy}{dx} = \dfrac{x\,\dfrac{d}{dx}(x^3+5x+2x^2+10) - (x^3+5x+2x^2+10)\dfrac{d}{dx}x}{x^2}$

$= \dfrac{x(3x^2+4x+5) - (x^3+2x^2+5x+10)}{x^2}$

9) $y = x - \dfrac{1}{x}$

$\dfrac{dy}{dx} = 1 + \dfrac{1}{x^2}$

10) $y = 5\tan x + 6\cot x$

$\dfrac{dy}{dx} = 5\sec^2 x - 6\operatorname{cosec}^2 x$

11) $y = \dfrac{x^2-1}{x^2+x+1}$

$\dfrac{dy}{dx} = \dfrac{(x^2+x+1)\dfrac{d}{dx}(x^2-1) - (x^2-1)\dfrac{d}{dx}(x^2+x+1)}{(x^2+x+1)^2}$

$= \dfrac{2x(x^2+x+1) - (2x+1)(x^2-1)}{(x^2+x+1)^2}$

(12) $y = \dfrac{1-\sin x}{1+\sin x}$

$$\frac{dy}{dx} = \frac{(1+\sin x)\frac{d}{dx}(1-\sin x) - (1-\sin x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(-\cos x)(1+\sin x) - \cos x(1-\sin x)}{(1+\sin x)^2}$$

$$= \frac{-\cos x - \cos x\sin x - \cos x + \sin x\cos x}{(1+\sin x)^2}$$

$$= \frac{-2\cos x}{(1+\sin x)^2}$$

(13) $y = x^2\cos x$

$$\frac{dy}{dx} = -x^2\sin x + 2x\cos x$$

(14) $y = \dfrac{1-\tan x}{1+\tan x}$

$$\frac{dy}{dx} = \frac{(1+\tan x)(-\sec^2 x) - \sec^2 x(1-\tan x)}{(1+\tan x)^2}$$

$$= \frac{-\sec^2 x - \sec^2 x\tan x - \sec^2 x + \sec^2 x\tan x}{(1+\tan x)^2}$$

$$= \frac{-2\sec^2 x}{(1+\tan x)^2}$$

(15) $y = \sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$

$= \sqrt{\dfrac{2\sin^2 x}{2\cos^2 x}}$

$= \sqrt{\tan^2 x}$

$= \tan x$

$\dfrac{dy}{dx} = \sec^2 x$

(16) $y = \sqrt{\dfrac{1-\sin 2x}{1+\sin 2x}}$

$= \sqrt{\dfrac{\sin^2 x + \cos^2 x - 2\sin x \cdot \cos x}{\sin^2 x + \cos^2 x + 2\sin x \cdot \cos x}}$

$= \sqrt{\dfrac{(\sin x - \cos x)^2}{(\sin x + \cos x)^2}}$

$= \dfrac{\sin x - \cos x}{\sin x + \cos x}$

$\dfrac{dy}{dx} = (\sin x + \cos x)(\cos x + \sin x) - (\sin x - \cos x)(\ldots$

(15) $y = \dfrac{a^x - b^x}{x}$

$\dfrac{dy}{dx} = \dfrac{x \frac{d}{dx}(a^x - b^x) - (a^x - b^x)\frac{d}{dx}x}{x^2}$

$= \dfrac{x(a^x \ln a - b^x \ln b) - (a^x - b^x)}{x^2}$

(17) $y = \dfrac{\left(x^{3/5} - 2e^{2\ln x} + \ln x^{2/3}\right)}{x+1}$

(18) $y = (2x^3 - 1)^4$

$\dfrac{dy}{dx} = 4(2x^3-1)\dfrac{d}{dx}(2x^3-1)$

$= 4(2x^3-1)(6x^2)$

(19) $y = (3x^2+2x+1)^8$

$\dfrac{dy}{dx} = 8(3x^2+2x+1)^7\dfrac{d}{dx}(3x^2+2x+1)$

$= 8(3x^2+2x+1)^7(6x+2)$

(20) $y = \left(2x^2 + \frac{3}{4}x + 7\right)^8$

$\dfrac{dy}{dx} = 8\left(2x^2 + \frac{3}{4}x + 7\right)^7 \dfrac{d}{dx}\left(2x^2 + \frac{3}{4}x + 7\right)$

$= 8\left(2x^2 + \frac{3}{4}x + 7\right)^7\left(4x + \frac{3}{4}\right)$

(21) $y = \left(\dfrac{2x^3+1}{3x^2+1}\right)^2$

$\dfrac{dy}{dx} = 2\left(\dfrac{2x^3+1}{3x^2+1}\right)\dfrac{d}{dx}\left(\dfrac{2x^3+1}{3x^2+1}\right)$

$= 2\left(\dfrac{2x^3+1}{3x^2+1}\right)\left[\dfrac{(3x^2+1)6x^2-(2x^3+1)6x}{(3x^2+1)^2}\right]$

(22) $y = (x^2+3)^4(x^2+5)^2$

$\dfrac{dy}{dx} = (x^2+3)^4\left[2(x^2+5)(2x)\right] + (x^2+5)^2\left[4(x^2+3)^3 \cdot 2x\right]$

(23) $y = (2x^4+x^2-x)^{1/3}$

$\dfrac{dy}{dx} = \dfrac{1}{3}(2x^4+x^2-x)^{-2/3}\dfrac{d}{dx}(2x^4+x^2-x)$

$= \dfrac{1}{3\sqrt[3]{2x^4+x^2-x}}(8x^3+2x-1)$

(24) $y = [\tan(3x^2+5)]^5$

$\dfrac{dy}{dx} = 5[\tan(3x^2+5)]^4\dfrac{d}{dx}[\tan(3x^2+5)]$

$= 5[\tan(3x^2+5)^4]\sec^2(3x^2+5)\cdot 6x$

(25) $y = \sqrt{\dfrac{1+\sin x}{1-\sin x}}$

$\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{1+\sin x}{1-\sin x}\right)^{-1/2}\dfrac{d}{dx}\left(\dfrac{1+\sin x}{1-\sin x}\right)$

$$= \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-1/2}\left[\frac{(1-\sin x)\cos x + (1+\sin x)\cos x}{(1-\sin)^2}\right]$$

23) $y = \sqrt{\tan x}$

$$\frac{dy}{dx} = \frac{\sec^2 x}{2\sqrt{\tan x}}$$

24) $y = \sqrt{1+\sin x}$

$$\frac{dy}{dx} = \frac{1}{2}(1+\sin x)^{-1/2} \cdot \cos x$$

25) $y = \left(\frac{2\tan x}{\tan x + \cos x}\right)^2$

$$\frac{dy}{dx} = 2\left(\frac{2\tan x}{\tan x+\cos x}\right)\frac{d}{dx}\left(\frac{2\tan x}{\tan x+\cos x}\right)$$

$$= 2\left(\frac{2\tan x}{\tan x+\cos x}\right)\left[\frac{(\tan x+\cos x)\frac{d}{dx}2\tan x - 2\tan x\frac{d}{dx}(\tan x+\cos x)}{(\tan x+\cos x)^2}\right.$$

$$= 2\left(\frac{2\tan x}{\tan x+\cos x}\right)\left[\frac{2\sec^2 x(\tan x+\cos x) - 2\tan x(\sec^2 x - \sin x)}{(\tan x+\cos x)^2}\right.$$

26) $y = \sqrt{\frac{1+x}{2+x}}$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{2+x}\right)^{-1/2}\left(\frac{(2+x)-(1+x)}{(2+x)^2}\right)$$

$$= \frac{1}{2\sqrt{\frac{1+x}{2+x}}}\left(\frac{2+x-1-x}{(2+x)^2}\right)$$

27)

$$y = \frac{1}{(x^3+\sin x)^2}$$

$$\frac{dy}{dx} = -2(x^3+\sin x)^{-3}\frac{d}{dx}(x^3+\sin x)$$

$$= \frac{-2}{\sqrt[3]{x^3+\sin x}}(3x^2+\cos x)$$

28)

$$y = \ln(\sqrt{x}+1)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}+1}\frac{d}{dx}\sqrt{x}+1$$

$$= \frac{1}{(\sqrt{x}+1)} \times \frac{1}{2\sqrt{x}}$$

29)

$$y = \sin 5x + \cos 7x$$

$$\frac{dy}{dx} = 5\cos 5x - 7\sin 7x$$

30)

$$y = \left(\frac{x+1}{x^2+3}\right)^{-3}$$

$$\frac{dy}{dx} = -3\left(\frac{x+1}{x^2+3}\right)^{-4}\left[\frac{(x^2+3)-(x+1)2x}{(x^2+3)^2}\right]$$

31)

$$y = e^{\sin^2 x}$$

$$\frac{dy}{dx} = e^{\sin^2 x}\frac{d}{dx}\sin^2 x$$

$$= e^{\sin^2 x} \times 2\sin x \cdot \cos x$$

$$= e^{\sin^2 x} \cdot \sin 2x$$

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32) $y = e^{\sin x^2}$

$\frac{dy}{dx} = e^{\sin x^2} \frac{d}{dx} \sin x^2$

$= e^{\sin x^2} \cdot \cos x^2 \cdot 2x$

33) $y = \log x^2$

$\frac{dy}{dx} = \frac{1}{x^2} \times 2x$

$= \frac{2}{x}$

34) $y = e^{2x} \sin 3x$

$\frac{dy}{dx} = e^{2x} \times 3\cos 3x + \sin 3x \cdot 2e^{2x}$

35) $y = \log\left(\frac{1-x}{1+x}\right)$

$\frac{dy}{dx} = \frac{1}{1-x/1+x} \frac{d}{dx}\left(\frac{1-x}{1+x}\right)$

$= \frac{1+x}{1-x}\left[\frac{1+x-1+x}{(1+x)^2}\right]$

$= \frac{1+x}{1-x} \times \frac{2x}{(1+x)^2}$

36) $y = \cos(\log x)^2$

$= 2\cos(\log x) \frac{d}{dx} \cos(\log x)$

$= \frac{2\cos(\log x) - \sin(\log x)}{x}$

(37) $y = \log(x + \sqrt{x^2 + a})$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2+a}} \frac{d}{dx}\left(x + \sqrt{x^2+a}\right)$$

$$= \frac{1}{x + \sqrt{x^2+a}} \left[1 + \frac{1}{2\sqrt{x^2+a}}\right] \times 2x$$

(38) $y = \frac{e^x \log x}{x^2}$

$$\frac{dy}{dx} = \frac{\frac{x^2 e^x}{x} - e^x \log x \cdot 2x}{x^4}$$

$$= \frac{x e^x - e^x \log x \, 2x}{x^4}$$

(39) $y = \sqrt{e^{\sqrt{x}}}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{e^{\sqrt{x}}}} \frac{d}{dx} e^{\sqrt{x}}$$

$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}} \cdot \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

(40) $y = \log(\log x)$

$$\frac{dy}{dx} = \frac{1}{\log x} \times \frac{1}{x}$$

(41) $y = \log(\sec x + \tan x)$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \times \sec x \cdot \tan x + \sec^2 x$$

42) $y = \log\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$

$$\frac{dy}{dx} = \frac{1-\sqrt{x}}{1+\sqrt{x}}\left(\frac{\frac{1-\sqrt{x}}{2\sqrt{x}} - \frac{1+\sqrt{x}}{2\sqrt{x}}}{(1-\sqrt{x})^2}\right)$$

$$= \frac{1-\sqrt{x}}{1+\sqrt{x}}\left[\frac{\frac{1-\sqrt{x}-1-\sqrt{x}}{2\sqrt{x}}}{(1-\sqrt{x})^2}\right]$$

$$= \frac{1-\sqrt{x}}{1+\sqrt{x}}\left[\frac{-1}{(1-\sqrt{x})^2}\right]$$

43) $y = \frac{\log x}{1+x\log x}$

$$\frac{dy}{dx} = \frac{1+x\log x}{x} - \log x\left(x\cdot 1 + \log x\right)$$

$$\frac{dy}{dx} = \frac{1+x\log x \times \frac{1}{x} - \log x\left(x/x + \log x\right)}{(1+x\log x)^2}$$

$$= \frac{\frac{1+x\log x}{x} - \log x(1+\log x)}{(1+x\log x)^2}$$

Inverse Trigonometry :-

Derivatives :-

Formulae:-

1. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

2. $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

3. $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

4. $\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$

5. $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$

6. $\frac{d}{dx}\operatorname{cosec}^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$

7. $1 - \cos x = 2\sin^2\frac{x}{2}$

8. $1 + \cos x = 2\cos^2\frac{x}{2}$

Note:-

1) Normal function.

2) Trigonometry with trigonometric function or more than one function.

Questions –

1. $\sin^{-1}2x$

2. $\tan^{-1}\sqrt{x}$

3. $\sqrt{\cot^{-1}\sqrt{x}}$

4. If $y = \frac{x\sin^{-1}x}{\sqrt{1-x^2}}$ then $\frac{dy}{dx}$.

Answers –

let,

$y = \sin^{-1}2x$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-4x^2}} \frac{d}{dx}2x$

$= \frac{2}{\sqrt{1-4x^2}}$

let,

$y = \tan^{-1}\sqrt{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x} \frac{d}{dx}\sqrt{x}$

$= \frac{1}{2\sqrt{x}(1+x)}$

let,

$y = \sqrt{\cot^{-1}\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\cot^{-1}\sqrt{x}}} \frac{d}{dx}\cot^{-1}\sqrt{x}$

$= -\frac{1}{2\sqrt{\cot^{-1}\sqrt{x}}} \times \frac{1}{1+x} \times \left(-\frac{1}{2\sqrt{x}}\right)$

$$y = \frac{x\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\Rightarrow y(\sqrt{1-x^2}) = x\sin^{-1}x$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x\sin^{-1}x}{\sqrt{1-x^2}}\right)$$

$$= \frac{\sqrt{1-x^2}\,\frac{d}{dx}\,x\sin^{-1}x - x\sin^{-1}x\,\frac{d}{dx}\sqrt{1-x^2}}{1-x^2}$$

$$= \frac{\sqrt{1-x^2}\left(\frac{x}{\sqrt{1-x^2}} + \sin^{-1}x\right) + \frac{2x}{2\sqrt{1-x^2}}\cdot x\sin^{-1}x}{1-x^2}$$

$$= \frac{x + (\sin^{-1}x)(\sqrt{1-x^2}) + \frac{x^2\sin^{-1}x}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{x + (\sin^{-1}x)(\sqrt{1-x^2}) + x^2\sin^{-1}x}{(1-x^2)^{3/2}}$$

Inverse derivative of trigonometric function:-

1. $\sin^2\left[\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right]$

let,

$$y = \sin^2\left[\cot^{-1}\frac{\sqrt{1+x}}{\sqrt{1-x}}\right]$$

Put, $x = \cos\theta$

$$= \sin^2\left[\cot^{-1}\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\right]$$

$$= \sin^2\left[\cot^{-1}\left(\sqrt{\frac{2\cos^2\theta/2}{2\sin^2\theta/2}}\right)\right]$$

$$= \sin^2\left[\cot^{-1}\left(\cot\theta/2\right)\right]$$

$$= \sin^2\theta/2$$

$$= \frac{1-\cos\theta}{2}$$

$$= \frac{1}{2} - \frac{x}{2}$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

2. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

let,

$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put,

$x = \tan\theta$

$= \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(\cos 2\theta)$

$\frac{dy}{dx} = \frac{d}{dx}\left[\cos^{-1}(\cos 2\theta)\right]$

$= \frac{d}{dx}\, 2\tan^{-1}x$

$= \frac{2}{1+x^2}$

3. $\sin^{-1}(3x - 4x^3)$

let,

$y = \sin^{-1}(3x - 4x^3)$

Put $x = \sin\theta$

$\theta = \sin^{-1}x$

$= \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$= \sin^{-1}(\sin 3\theta)$

$= 3\theta$ $\quad = 3 \times \sin^{-1}x$

$dy/dx = 3\theta$ $\quad = \frac{3}{\sqrt{1-x^2}}$

5. $\tan^{-1}\left(\frac{4\sqrt{x}}{1-4x}\right)$

let,

$$y = \tan^{-1}\left(\frac{4\sqrt{x}}{1-4x}\right)$$

$$= \tan^{-1}\left(\frac{2\sqrt{x}+2\sqrt{x}}{1-2\sqrt{x}\cdot 2\sqrt{x}}\right)$$

$$= \tan^{-1}2\sqrt{x} + \tan^{-1}2\sqrt{x}$$

$$= \frac{1}{1+4x} + \frac{1}{1+4x}$$

$$= \frac{2}{1+4x}$$

$$= \frac{1}{1+4x}\frac{d}{dx}2\sqrt{x} + \frac{1}{1+4x}\frac{d}{dx}2\sqrt{x}$$

$$= \frac{1}{\sqrt{x}(1+4x)} + \frac{1}{\sqrt{x}(1+4x)}$$

$$= \frac{2}{\sqrt{x}(1+4x)}$$

$$= \frac{2}{\sqrt{x}+4x\sqrt{x}} \text{ (Ans)}$$

let

$$y = \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2 x/2}{2\sin x/2 \cos x/2}\right)$$

$$= \tan^{-1}\left(\tan x/2\right)$$

$$= x/2$$

$$\frac{dy}{dx} = \frac{1}{2}$$

let

$$y = \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$$

$$= \tan^{-1}\left(\frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}}\right)$$

$$= \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$= \tan^{-1}\left(\frac{1-1\cdot\tan x}{1-1\cdot\tan x}\right)$$

$$= \tan^{-1}\left(\frac{\pi/4 - \tan x}{1 - \pi/4 \cdot \tan x}\right)$$

$$= \tan^{-1}\left(\tan\left(\pi/4 - x\right)\right)$$

$$= \pi/4 - x$$

$$\frac{dy}{dx} = -1$$

3) $y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

$= \tan^{-1}\left(\sqrt{\frac{2\sin^2 x/2}{2\cos^2 x/2}}\right)$

$= \tan^{-1}(\tan x/2)$

$= \frac{x}{2}$

$= \frac{1}{2}$

4) $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$= \tan^{-1}\left(\frac{\cos^2 x/2 - \sin^2 x/2}{(\cos x/2 + \sin x/2)^2}\right)$

$= \tan^{-1}\left[\frac{(\cos x/2 + \sin x/2)(\cos x/2 - \sin x/2)}{(\cos x/2 + \sin x/2)^2}\right]$

$= \tan^{-1}\left[\frac{\cos x/2 - \sin x/2}{\cos x/2 + \sin x/2}\right]$

$= \tan^{-1}\left[\frac{1 - \tan x/2}{1 + \tan x/2}\right]$

$= \tan^{-1}\left[\tan(\pi/4 - x/2)\right]$

$= -x/2$

$= -1/2$

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$\{\tan^{-1}(\sec x + \tan x)\}$

let,

$$y = \tan^{-1}(\sec x + \tan x)$$

$$= \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)$$

$$= \tan^{-1}\left(\frac{1 + \sin x}{\cos x}\right)$$

$$= \tan^{-1}\left[\frac{(\cos x/2 + \sin x/2)^2}{\cos^2 x/2 - \sin^2 x/2}\right]$$

$$= \tan^{-1}\left[\frac{(\cos x/2 + \sin x/2)^2}{(\cos x/2 - \sin x/2)(\cos x/2 + \sin x/2)}\right]$$

$$= \tan^{-1}\left[\frac{1 + \tan x/2}{1 - \tan x/2}\right]$$

$$= \tan^{-1}\left(\pi/4 + x/2\right)$$

$$= \tan^{-1}$$

$$= \tan^{-1}\left[\tan\left(\pi/4 + x/2\right)\right]$$

$$\frac{dy}{dx} = \frac{1}{2}$$

let,

$$y = \left\{\cos^{-1}\sqrt{\frac{1 + \cos x}{2}}\right\}$$

$$= \cos^{-1}\left(\sqrt{\frac{2\sin^2 x/2}{2}}\right)$$

$$= \cos^{-1}(\cos x/2)$$

$$= x/2$$

$$dy/dx = 1/2$$

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$$= \tan^{-1}\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\theta/2}}$$

$$= \tan^{-1}(\tan\theta/2)$$

$$= \theta/2$$

$$= \frac{\cos^{-1}x}{2}$$

$$\frac{dy}{dx} = -\frac{1}{2\sqrt{1-x^2}}$$

3. $(\tan^{-1}5x)^2$

let,

$$y = (\tan^{-1}5x)^2$$

$$\frac{dy}{dx} = 2\tan^{-1}5x \times \frac{5}{1+25x^2}$$

4. $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

let,

$$y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$$

$$= \tan^{-1}\left(\frac{\cos^2 x/2 - \sin^2 x/2}{(\cos x/2 + \sin x/2)^2}\right)$$

$$= \tan^{-1}\left(\frac{\cos x/2 - \sin x/2}{\cos x/2 + \sin x/2}\right)$$

$= \tan^{-1}\left(\frac{1 - \tan x/2}{1 + \tan x/2}\right)$

$= \tan^{-1}\left[\tan\left(\pi/4 - x/2\right)\right]$

$= \pi/4 - x/2$

$\frac{dy}{dx} = -\frac{1}{2}$

5. $\tan^{-1}x - \cot^{-1}x$

let,

$y = \tan^{-1}x - \cot^{-1}x$

$\frac{dy}{dx} = \frac{1}{1+x^2} + \frac{1}{-1+x^2}$

$= \frac{2}{1+x^2}$

6. $\sin^{-1}2ax\sqrt{1-a^2x^2}$

let,

$y = \sin^{-1}2ax\sqrt{1-a^2x^2}$

put,

$x = \sin\theta$

$\theta = \sin^{-1}x$

$= \sin^{-1}2 \times a\sin\theta\sqrt{1 - a^2\sin^2\theta}$

$= \sin^{-1}(2a\sin\theta \cdot a\cos\theta)$

$= \sin^{-1}(a\sin 2\theta)$

$= 2a\theta$

$= 2a\sin^{-1}x$

$\frac{dy}{dx} = \frac{2a}{\sqrt{1-4a^2}}$

7. $\sin^{-1}\left(2ax\sqrt{1-a^2x^2}\right)$

let,

$y = \sin^{-1}\left(2ax\sqrt{1-(ax)^2}\right)$

Put,

$ax = \sin\theta$

$\Rightarrow \theta = \sin^{-1}ax$

$\therefore y = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$= \sin^{-1}(2\sin\theta\cdot\cos\theta)$

$= \sin^{-1}(\sin 2\theta)$

$= 2\theta$

$= 2\sin^{-1}ax$

$\frac{dy}{dx} = \frac{2a}{\sqrt{1-a^2x^2}}$

$y = \sec^{-1}\left(\frac{\sqrt{x^2+a^2}}{a}\right)$

Put,

$x = a\tan\theta$

$\Rightarrow \theta = a\tan^{-1}x$

$= \sec^{-1}\left(\frac{\sqrt{a^2\tan^2\theta + a^2}}{a}\right)$

$$= \sec^{-1}\left(\frac{\sqrt{a^2(\tan\theta+1)}}{a}\right)$$

$$= \sec^{-1}\left(\frac{\sqrt{a^2\sec^2\theta}}{a}\right)$$

$$= \sec^{-1}\left(\frac{a\sec\theta}{a}\right)$$

$$= \theta$$

$$= a\tan^{-1}x$$

$$= \frac{a}{1+x^2}$$

Exercise :—

1) $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$

differentiate x and y with respect to 't'

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$$

$$= \frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{-2t(1+t^2) - 2t(1-t^2)}{(1+t^2)^2}$$

$$= \frac{-2t[(1+t^2)+(1-t^2)]}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$

$$= \frac{2(1+t^2) - 4t^2}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{\dfrac{2(1+t^2) - 4t^2}{(1+t^2)^2}}{\dfrac{-2t\left[(1+t^2) + (1-t^2)\right]}{(1+t^2)^2}}$$

$$= \frac{2 + 2t^2 - 4t^2}{-2t(1+t^2) + (1-t^2)}$$

$$= \frac{2 - 2t^2}{-2t(1+t^2) + (1-t^2)}$$

$x = a\cos\theta$ and $y = b\sin\theta$

differentiating x and y with respect to 'θ'

$$\frac{dx}{d\theta} = -a\sin\theta$$

$$\frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = -\frac{b}{a}\cot\theta$$

3) $x = a\sin^3 t$ and $y = b\cos^3 t$

differentiating x and y with respect to 't'

$$\frac{dx}{dt} = 3a\sin^2 t \cdot \cos t$$

$$\frac{dy}{dt} = -3b\cos^2 t \cdot \sin t$$

$$\frac{dy}{dx} = \frac{-3b\cos^2 t \cdot \sin t}{3a\sin^2 t \cdot \cos t}$$

$$= -\frac{b}{a}\cot\theta$$

4) $x = a\left(\theta + \frac{1}{\theta}\right)$ and $y = a\left(\theta - \frac{1}{\theta}\right)$

differentiating x and y with respect to 'θ'

$$\frac{dx}{d\theta} = a\left(1 - \frac{1}{\theta^2}\right) = a\left(\frac{\theta^2 - 1}{\theta^2}\right)$$

$$\frac{dy}{d\theta} = a\left(1 + \frac{1}{\theta^2}\right) = a\left(\frac{\theta^2 + 1}{\theta^2}\right)$$

$$\frac{dy}{dx} = \frac{a\left(\frac{\theta^2 + 1}{\theta^2}\right)}{a\left(\frac{\theta^2 - 1}{\theta^2}\right)}$$

$$= \frac{a}{a}\left(\frac{\theta^2 + 1}{\theta^2 - 1}\right)$$

5) $x = \frac{2at}{1+t^2}$ and $y = \frac{2bt}{1-t^2}$

differentiating x and y with respect to 't'

$$\frac{dx}{dt} = \frac{2a(1+t^2) - 4at^2}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2b(1-t^2) + 4bt^2}{(1-t^2)^2}$$

$$\frac{dy}{dx} = \frac{\dfrac{2b(1-t^2) + 4bt}{(1-t^2)^2}}{\dfrac{2a(1+t^2) - 4at}{(1+t^2)^2}}$$

Q) $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$

Differentiating x and y with respect to 'θ'

$$\frac{dx}{d\theta} = a(1 + \cos\theta) = 2a\cos^2\theta/2$$

$$\frac{dy}{d\theta} = a\sin\theta = 2a\sin\theta/2 \cdot \cos\theta/2$$

$$\frac{dy}{dx} = \frac{2a\sin\theta/2 \cdot \cos\theta/2}{2a\cos^2\theta/2}$$

$$= \tan\theta/2$$

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Q- $\frac{dy}{dx} = ?$

If $\sin x = \frac{2t}{1+t^2}$

$\tan y = \frac{2t}{1-t^2}$

$\sin x = \frac{2t}{1+t^2}$

$\Rightarrow x = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$

- Put,

$t = \tan\theta$

$\Rightarrow \theta = \tan^{-1} t$

$= \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \sin^{-1}(\sin 2\theta)$

$= 2\theta$

$= 2\tan^{-1} t$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$\tan y = \frac{2t}{1-t^2}$

$\Rightarrow y = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$

- put

$t = \tan\theta$

$\Rightarrow \theta = \tan^{-1} t$

$$= \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1}t$$

$$\frac{dy}{dt} = \frac{2}{1+t^2}$$

$$\frac{dy}{dx} = \frac{2}{1+t^2} \Big/ \frac{2}{1+t^2}$$

$$= 1 \text{ (Ans)}$$

7) $x = at^2$ and $y = at^3$

differentiating x and y w.r.t 't'

$$\frac{dx}{dt} = 2at$$

$$\frac{dy}{dt} = 3at^2$$

$$\frac{dy}{dx} = \frac{3at^2}{2at}$$

$$= \frac{3}{2}t$$

2) $x = \frac{2at}{1+t^2}$ and $y = \frac{2bt}{1-t^2}$

Differentiating x and y w.r.t. 't'

$$\frac{dx}{dt} = \frac{2a(1+t^2) - 4at^2}{(1+t^2)^2} = \frac{2a + 2at^2 - 4at^2}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2b(1-t^2) - 4bt^2}{(1-t^2)^2} = \frac{2b + 2bt^2 + 4bt^2}{(1-t^2)^2}$$

$\frac{dy}{dx}$

$$\frac{dx}{dy} = \frac{\frac{2a + 2at^2 - 4at^2}{(1+t^2)^2}}{\frac{2b + 2bt^2 + 4bt^2}{(1+t^2)^2}}$$

$$= \frac{\frac{2a - 2at^2}{(1+t^2)^2}}{\frac{2b + 2bt^2}{(1-t^2)^2}}$$

$$= \frac{2a(1-t^2)}{(1+t^2)^2} \times \frac{(1-t^2)^2}{2b(1+t^2)}$$

$$= \frac{a(1-t^2)^3}{b(1+t^2)^3}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{b(1+t^2)^3}{a(1-t^2)^3}.$$

They are not very ~~accurate~~ accurate.

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8) $x = \frac{at^2}{1+t^2}$ and $y = \frac{at^3}{1+t^2}$

Differentiating x and y w.r.t 't'

$$\frac{dx}{dt} = \frac{2at(1+t^2) - 2at^3}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{3at^2(1+t^2) - 2at^4}{(1+t^2)^2}$$

$$\frac{dy}{dx} = \frac{3at^2 + 3at^4 - 2at^4}{2at + 2at^3 - 2at^3}$$

$$= \frac{2at}{3at^2 + at^4} \qquad = \frac{3at^2 + t^4}{2at}$$

$$= \frac{2at}{at^2(3+t^2)} \qquad = \frac{at^2(3+t^2)}{2at}$$

$$= \frac{2}{a(t^2+3)} \qquad = \frac{a(3+t^2)}{2}$$

9) $x = \frac{a(1-t)}{1+t^2}$ and $y = at\left(\frac{1-t^2}{1+t^2}\right)$

Differentiating 'x' and 'y' with respect to 't'

$$x = \frac{a(1-t)}{1+t^2}$$

$$= \frac{a - at}{1+t^2}$$

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{a-at}{1+t^2}\right)$$

$$= \frac{(1+t^2)\frac{d}{dt}(a-at) - (a-at)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{-a(1+t^2) - 2t(a-at)}{(1+t^2)^2}$$

$$y = at\left(\frac{1-t^2}{1+t^2}\right)$$

$$= \frac{at - at^3}{1+t^2}$$

$$\frac{dy}{dt} = \frac{(1+t^2)\frac{d}{dt}(at - at^3) - (at - at^3)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{(1+t^2)(a - 3at^2) - 2t(at - at^3)}{(1+t^2)^2}$$

$$= \frac{(a - 3at^2)(1 + t^2) - 2t(at - at^3)}{(1 + t^2)^2}$$

$$\frac{dy}{dx} = \frac{\frac{(a - 3at^2)(1 + t^2) - 2t(at - at^3)}{(1 + t^2)^2}}{-a(1 + t^2) - 2t(a - at)}$$

$$= \frac{a + at^2 - 3at^2 - 3at^4 - 2t^2 + 2at^4}{-a - at^2 - 2at + 2at^2}$$

$$= \frac{a - at^2 - 2at^2 - at^4}{-(a + 2at + at^2)}$$

[illegible]

$$= \frac{a - 2at^2 - 2t - at^4}{-a(1 + 2t + t^2)}$$

$$= \frac{a(1 - 2t^2) - t(2 - t^3)}{-a(1 + 2t + t^2)}$$

11. $ax^2 + by^2 = 25$

differentiating both the sides w.r.t 'x'

$\Rightarrow 2ax + 2b\frac{dy}{dx} = 0$

$\Rightarrow 2by\frac{dy}{dx} = -2ax$

$\Rightarrow \frac{dy}{dx} = -\frac{2ax}{2b}$

$= -\frac{ax}{by}$

12. $3x^2 + 3y^2 + 4x - 3y + 12 = 0$

differentiating both the sides w.r.t 'x'

$\Rightarrow 6x + 6y\frac{dy}{dx} + 4 - 3\frac{dy}{dx} = 0$

$\Rightarrow 6x + 4 + 6y\frac{dy}{dx} - 3\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx}(6y - 3) = -(6x + 4)$

$\Rightarrow \frac{dy}{dx} = -\frac{(6x+4)}{6y+3}$

$= -\frac{2(3x+2)}{3(2y+1)}$

$=$

13. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

differentiating both the sides w.r.t 'x'

$\Rightarrow \frac{18x}{9} + \frac{32y}{16} = 0$

$\Rightarrow \frac{18x \times 16 + 32y \times 9}{144} = 0$

14. $x^2 + y^2 + 5xy = 0$

Differentiating b.t.s w.r.t 'x'

$\Rightarrow 2x + 2y\frac{dy}{dx} + 5\left(x\frac{dy}{dx} + y\right) = 0$

$\Rightarrow 2x + 2y\frac{dy}{dx} + 5\frac{dy}{dx} + 5y = 0$

$\Rightarrow \frac{dy}{dx}(2y + 5) = -(2x + 5y)$

$\Rightarrow \frac{dy}{dx} = -\frac{(2x + 5y)}{(2y + 5x)}$

15. $e^{xy} + y\sin x = 1$

Differentiating both the sides w.r.t 'x'

$\Rightarrow e^{xy}\left(x\frac{dy}{dx} + y\right) + y\cos x + \sin x\frac{dy}{dx} = 0$

$\Rightarrow xe^{xy}\frac{dy}{dx} + ye^{xy} + y\cos x + \sin x\frac{dy}{dx} = 0$

$\Rightarrow (xe^{xy} + \sin x)\frac{dy}{dx} + ye^{xy} + y\cos x = 0$

$\Rightarrow \frac{dy}{dx}(xe^{xy} + \sin x) = -(ye^{xy} + y\cos x)$

$\Rightarrow \frac{dy}{dx} = -\frac{(ye^{xy} + y\cos x)}{xe^{xy} + \sin x}$

16. $\ln\sqrt{x^2+y^2} = \tan^{-1}\left(\frac{y}{x}\right)$

Differentiating both the sides with respect to 'x'

$\Rightarrow \frac{1}{\sqrt{x^2+y^2}} \frac{d}{dx}\sqrt{x^2+y^2} = \frac{x^2}{x^2+y^2} \frac{d}{dx}\left(\frac{y}{x}\right)$

$\Rightarrow \frac{1}{2(x^2+y^2)} \frac{d}{dx}(x^2+y^2) = \frac{x^2}{x^2+y^2}\left[\frac{x\frac{dy}{dx} - y}{x^2}\right]$

$\Rightarrow \frac{1}{2(x^2+y^2)} \times 2x + 2y\frac{dy}{dx} = \frac{x^2}{x^2+y^2} \times \frac{x\frac{dy}{dx} - y}{x^2}$

$\Rightarrow \frac{2(x + y\frac{dy}{dx})}{2(x^2+y^2)} = \frac{x\frac{dy}{dx} - y}{x^2+y^2}$

$\Rightarrow x + y\frac{dy}{dx} = x\frac{dy}{dx} - y$

$\Rightarrow (y - x)\frac{dy}{dx} = -(x+y)$

$\Rightarrow \frac{dy}{dx} = \frac{-(x+y)}{-(x-y)}$

$= \frac{x+y}{}$

Differentiation

Standard formula

① $\frac{d}{dx}(x)^n = n \cdot x^{n-1}$

② $\frac{d}{dx} a^x = a^x \ln a \quad , \quad a > 0 \quad , a \neq 1$

③ $\frac{d}{dx} e^x = e^x$

④ $\frac{d}{dx} \log_a x = \frac{1}{x \ln a} \quad , \quad x > 0 \ , \ a > 0 \ , \ a \neq 1$

⑤ $\frac{d}{dx} \log x = \frac{1}{x}$

⑥ $\frac{d}{dx}(k) = 0$

⑦ $\frac{d}{dx} \sin x = \cos x$

⑧ $\frac{d}{dx} \cos x = -\sin x$

⑨ $\frac{d}{dx} \tan x = \sec^2 x$

⑩ $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$

⑪ $\frac{d}{dx} \sec x = \sec x \cdot \tan x$

⑫ $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x$

① $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2} \quad , \quad x \neq 0$

② $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} \quad , \quad x > 0$

③ $\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1} \times a$

④ $\frac{d}{dx} a^{mx} = a^{mx} (\ln a) m$

⑤ $\frac{d}{dx} e^{mx} = e^{mx} \times m$

⑥ $\frac{d}{dx} \sin \quad = (\cos Mx) \times m$

$\frac{d}{dx} e^{2\ln x} = \frac{d}{dx} x^2 = 2x$

$\frac{d}{dx} 3^{\log_3 x} = \frac{d}{dx} x = 1$

$\frac{d}{dx} 9^{\log_3 x} = \frac{d}{dx}(3^2)^{\log_3 x}$

$= \frac{d}{dx} 3^{2\log_3 x} = \frac{d}{dx} x^2$

$= 2x$

$\frac{d}{dx} 3^{\log_9 x} = \frac{d}{dx} (3^{1/2})^{\log_3 x}$

$= \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2}$

$= \frac{1}{2\sqrt{x}}$

Q. Find $\frac{d}{dx} e^{[x] \ln(1+x)}$ where $1 \le x < 2$

$\frac{d}{dx} e^{[x] \ln(1+x)}$

$= \frac{d}{dx} e^{\ln(1+x)}$

$= \frac{d}{dx} (1+x)^2$

$= 2(1+x)^2$

Q. Find $\frac{d}{dx} e^{\ln|x|}$, where $x < 0$

$\frac{d}{dx} e^{\ln|x|} = \frac{d}{dx}$

$= \frac{d}{dx} |x|$

$= \frac{d}{dx}(-x)$

$= -1$

Find $[x]$ at $x = 2$

Not differentiable

Algebra of derivatives:-

① $\frac{d}{dx}(u+v) = \frac{d}{dx} u \pm \frac{dv}{dx}$

② $\frac{d}{dx} k f(x) = k \cdot \frac{d}{dx} f(x)$

③ $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

④ $\frac{d}{dx}(uvw) = uv \frac{dw}{dx} + uw \cdot \frac{dv}{dx} + vw \frac{du}{dx}$

If $f(x) = (x-1)(x-2)(x-3) \cdots (x-107)$ the $f'(1) = ?$

$f'(x) = (x-1)(x-2) \cdots \frac{d}{dx}(x-107) + (x-1)(x-2) \cdots (x-105)$

$(x-107) \frac{d}{dx}(x-106) + \cdots + (x-2)(x-3) \cdots (x-107) \frac{d}{dx}(x-1)$

$f'(1) = (-1)(-2) \cdots (-106) \times 1$

$= 106!$

Q. $y = {}^{15}C_0 + {}^{15}C_1 x + {}^{15}C_2 x^2 + \cdots + {}^{15}C_{15} x^{15}$

$y = {}^{15}C_0 + {}^{15}C_1 x + {}^{15}C_2 x^2 + \cdots + {}^{15}C_{15} x^{15}$

$= (1+x)^{15}$

$\frac{dy}{dx} = 15 (1+x)^{14}$

Derivative of composite functions

Chain rule

$y = (x+1)^5 \Rightarrow y = t^5 \Rightarrow \frac{d}{dt}(y)$

$\Rightarrow \frac{d}{dt}(t^5) \Rightarrow \frac{dy}{dt} = 5t^4$

$\Rightarrow \frac{dy}{dt} \; 5(x+1)^4$

Q. $\frac{d}{dx} (x^2+5)^8$

$\frac{d}{dx} (x^2+5)^8 = 8(x^2+5)^7 \cdot \frac{d}{dx}(x^2+5)$

$= 8(x^2+5)^7 \cdot 2x$

Q. $\frac{d}{dx} \frac{1}{(x^3+\sin x)^2}$

$\frac{d}{dx} (x^3+\sin x)^{-2} = -2(x^3+\sin x)^{-3} \cdot (3x^2 + \cos x)$

Q. $\frac{d}{dx} \ln(\sqrt{x}+1)$

$\frac{1}{\sqrt{x}+1} \cdot \frac{1}{2\sqrt{x}}$

Q. $\frac{d}{dx} \sin 5x + \cos 7x$

$= 5\cos 5x - 7 \sin 7x$

Q. $\frac{d}{dt} e^{\sin t}$

$e^{\sin t} \cdot \cos t$

Q. $\frac{d}{dx} \sqrt{ax^2+bx+c}$

$= \sqrt{ax^2+bx+c} \cdot \frac{d}{dx}(ax^2+bx+c)$

$= \frac{1}{2\sqrt{ax^2+bx+c}} \left[a \times 2x + b \times 1 + 0\right]$

$= \frac{2nx + b}{2\sqrt{nx^2+bx+c}}$

Q. $\frac{d}{dx}\left(\frac{x^2+3}{x+1}\right)^3$

$= 3\left(\frac{x^2+3}{x+1}\right)^2 \cdot \frac{d}{dx}\left(\frac{x^2+3}{x+1}\right)$

$= \frac{3(x^2+3)^2}{(x+1)^2}\left[\frac{(x+1)\frac{d}{dx}(x^2+3) - (x^2+3)\frac{d}{dx}(x+1)}{(x+1)^2}\right]$

$= \frac{3(x^2+3)^2}{(x+1)^4}\left[(x+1)2x - (x^2+3)\right]$

$= \frac{3(x^2+3)^2}{(x+1)^4}\left[2x^2 + 2x - x^2 - 3\right]$

Q. $\frac{d}{d\theta}\sec(\tan\theta)$

$\sec(\tan\theta)\tan(\tan\theta)\cdot\sec^2\theta$

Q. $\frac{d}{dx}\sin\left(\frac{1-x^2}{1+x^2}\right)$

$= \cos\left(\frac{1-x^2}{1+x^2}\right)\cdot\left[\frac{(1+x^2)\frac{d}{dx}(1-x^2) - (1-x^2)\frac{d}{dx}(1+x^2)}{(1+x^2)^2}\right]$

$= \cos\left(\frac{1-x^2}{1+x^2}\right)\cdot\left[\frac{(1+x^2)\cdot(-2x) - (1-x^2)\cdot 2x}{(1+x^2)^2}\right]$

$= \cos\left(\frac{1-x^2}{1+x^2}\right)\left[\frac{-2x(1+x^2+1-x^2)}{(1+x^2)^2}\right]$

$= \cos\left(\frac{1-x^2}{1+x^2}\right)\frac{(-4x)}{(1+x^2)^2}$

Q. $\frac{d}{dx}\sqrt{\tan 3x}$

$= \frac{1}{2\sqrt{\tan 3x}}\cdot\sec^2 3x\cdot 3$

$= 3\sec^2 3x / 2\sqrt{\tan 3x}$

Q. $\frac{d}{dx}\tan^3 x$

Q. P.T. $\frac{d}{dx}\left[\frac{1-\tan x}{1+\tan x}\right]^{1/2} = \frac{-1}{\sqrt{\cos 2x}\,[\cos x + \sin x]}$

Sol. $\frac{d}{dx}\left[\frac{1-\tan x}{1+\tan x}\right]^{1/2} = \frac{1}{2}\left[\frac{1-\tan x}{1+\tan x}\right]^{-1/2}\frac{d}{dx}\left[\frac{1-\tan x}{1+\tan x}\right]$

$= \frac{1}{2}\left(\frac{1+\tan x}{1-\tan x}\right)^{1/2}\left[\frac{(1+\tan x)\frac{d}{dx}(1-\tan x) - (1-\tan x)\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}\right]$

$= \frac{1}{2}\frac{\sqrt{1+\tan x}}{\sqrt{1-\tan x}}\left[\frac{(1+\tan x)(-\sec^2 x) - (1-\tan x)\sec^2 x}{\sqrt{1+\tan x}\,(1+\tan x)}\right]$

$= \frac{-\sec^2 x\,[1+\tan x + 1 - \tan x]}{2\sqrt{1-\tan x}\,(1+\tan x)}$

$= \frac{-2\;1/\cos^2 x}{2\sqrt{1-\frac{\sin x}{\cos x}}\left(1+\frac{\sin x}{\cos x}\right)} = \frac{-1/\cos^2 x}{\frac{\sqrt{\cos^2 x - \sin^2 x}}{\cos x}\left(\frac{\cos x + \sin x}{\cos x}\right)}$

$= \frac{-1}{\sqrt{\cos 2x}\,(\cos x + \sin x)}$

Derivative of Inverse trigonometric functions:-

① $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$

proof. Let $y = \sin^{-1}x \Rightarrow \sin y = x$

$\Rightarrow \frac{d}{dx}(\sin y) = \frac{d}{dx}x \Rightarrow \cos y\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

$= \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

② $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$, $-1 < x < 1$

proof. Let $y = \cos^{-1}x \Rightarrow \cos y = x$

$\Rightarrow \frac{d}{dx}\cos y = \frac{d}{dx}x \Rightarrow -\sin y\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y} \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$

(3) $\frac{d}{dx}\tan^{-1}x$

Proof: Let $y = \tan^{-1}x$, $x \in R$

$\tan y = x \Rightarrow \frac{d}{dx}\tan y = \frac{d}{dx}x$

$\Rightarrow \sec^2 y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} \Rightarrow \frac{dy}{dx} = \frac{1}{1+\tan^2 y}$

$= \frac{1}{1+x^2}$

(4) $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$, $x \in R$

Proof: Let $y = \cot^{-1}x$, $x \in R$

$\Rightarrow \cot y = x \Rightarrow \frac{d}{dx}\cot y = \frac{d}{dx}x$

$\Rightarrow -\operatorname{cosec}^2 y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{-1}{\operatorname{cosec}^2 y}$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2}$, $x \in R$

(5) $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$, $x \in R - [-1, 1]$

Proof: Let $y = \sec^{-1}x$, $x \in R$

$\Rightarrow \sec y = x \Rightarrow \frac{d}{dx}\sec y = \frac{d}{dx}x$

$\Rightarrow \tan y \sec y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y}$

$= \frac{1}{\sec y\sqrt{\sec^2 y - 1}}$

$= \frac{1}{|x|\sqrt{x^2-1}}$

Ex. $\frac{d}{dx}\sin^{-1}2x = \frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}2x$

$= \frac{2}{\sqrt{1-4x^2}}$

Ex. $\frac{d}{dx}\cot^{-1}\sqrt{x} = \frac{-1}{1+(\sqrt{x})^2}\frac{d}{dx}\sqrt{x}$

$= \frac{-1}{1+x} \times \frac{1}{2\sqrt{x}} = \frac{-1}{(1+x)(2\sqrt{x})}$

Ex. $\frac{d}{dx}\sec^{-1}(2x+1) = \frac{1}{|2x+1|\sqrt{(2x+1)^2-1}}\frac{d}{dx}(2x+1)$

$= \frac{2}{|2x+1|\sqrt{(2x+1)^2-1}} = \frac{2}{|2x+1|\sqrt{4x^2+4x+1-1}}$

$= \frac{2}{|2x+1|\sqrt{4(x^2+x)}} = \frac{2}{2|2x+1|\sqrt{x^2+x}}$

Ex. $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} = \begin{cases} \frac{1}{x\sqrt{x^2-1}}, & x > 1 \\ \frac{1}{-x\sqrt{x^2-1}}, & x < -1 \end{cases}$

Ex. $\frac{d}{dx}\operatorname{cosec}^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$

$= \begin{cases} \frac{-1}{x\sqrt{x^2-1}}, & x > 1 \\ \frac{1}{x\sqrt{x^2-1}}, & x < -1 \end{cases}$

Derivative of inverse trigonometry functions by proper substitution

General substitution form:-

$\sqrt{1-x^2}$, $x = \sin\theta$ or $\cos\theta$

$\sqrt{1+x^2}$, $x = \tan\theta$ or $\cot\theta$

$\sqrt{x^2-1}$, $x = \sec\theta$ or $\operatorname{cosec}\theta$

$\frac{2x}{1+x^2}$, $x = \tan\theta$

$\frac{1-x^2}{1+x^2}$, $x = \tan\theta$

$1-2x^2$, $x = \sin\theta$

$2x^2-1$, $x = \cos\theta$

$\sqrt{1\pm x}$, $x = \cos 2\theta$

$\sqrt{1+\cos 2\theta} = \sqrt{2}\cos\theta$

$3x-4x^3$, $x = \sin\theta$

$4x^3-3x$, $x = \cos\theta$

$\sqrt{1-\sin^2\theta} = \cos\theta$

$\sqrt{1+\tan^2\theta} = \sec\theta$

$\sqrt{\sec^2\theta-1} = \tan\theta$

$\frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$

$\frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$

$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

$1-2\sin^2\theta = \cos 2\theta$

$2\cos^2\theta - 1 = \cos 2\theta$

$\sqrt{1+\cos 2\theta} = \sqrt{2}\cos\theta$

$\sqrt{1-\cos 2\theta} = \sqrt{2}\sin\theta$

$3\sin\theta - 4\sin^3\theta = \sin 3\theta$

$4\cos^3\theta - 3\cos\theta = \cos 3\theta$

Q $\frac{d}{dx} \sin^{-1} 2x\sqrt{1-x^2}$, let $x = \sin\theta$

$\Rightarrow \frac{d}{dx} \sin^{-1} 2\sin\theta\sqrt{1-\sin^2\theta}$

$\Rightarrow \frac{d}{dx} \sin^{-1} 2\sin\theta\cos\theta$

$= \frac{d}{dx} \sin^{-1} \sin 2\theta = \frac{d}{dx} 2\theta = 2\frac{d}{dx}\theta$

$= 2 \times \frac{d}{dx} \sin^{-1} x$

$= 2 \times \frac{1}{\sqrt{1-x^2}}$

Q $\frac{d}{dx} \tan^{-1} \frac{2x}{1-x^2}$ $x = \tan\theta$

$= \frac{d}{dx} \tan^{-1} \frac{\tan 2\theta}{1-(\tan 2\theta)^2}$

$= \frac{d}{dx} \tan^{-1} \tan 2\theta$

$= \frac{d}{dx} 2\theta = 2\frac{d}{dx}\theta \Rightarrow 2\frac{d}{dx}\tan^{-1} x$

$\Rightarrow 2 \times \frac{1}{1+x^2}$

Q $\frac{d}{dx} \sin^{-1} \frac{2x}{1+x^2}$ let $x = \tan 2\theta$

$= \frac{d}{dx} \sin^{-1} \frac{\tan 2\theta}{1+(\tan 2\theta)^2}$

$= \frac{d}{dx} \sin^{-1} \sin 2\theta$

$= \frac{d}{dx} 2\theta = 2\frac{d}{dx}\theta = 2\frac{d}{dx}\tan^{-1} x$

$= 2 \times \frac{1}{1+x^2}$

Q $\frac{d}{dx} \cos^{-1} \sqrt{\frac{1+x}{2}}$ $x = \cos 2\theta$

$= \frac{d}{dx} \cos^{-1} \sqrt{\frac{1+\cos 2\theta}{2}}$ $\Rightarrow \cos^{-1} x = 2\theta$ $\Rightarrow \frac{1}{2}\cos^{-1} x = \theta$

$= \frac{d}{dx} \cos^{-1} \sqrt{\frac{2\cos^2\theta}{2}}$

$\frac{d}{dx} \sin^{-1} 2x\sqrt{1-x^2}$

let $x = \cos\theta$

$\frac{d}{dx} \sin^{-1} 2\cos\theta\sqrt{1-\cos^2\theta}$

$= \frac{d}{dx} \sin^{-1} 2\cos\theta \sin\theta$

$= \frac{d}{dx} \sin^{-1} \sin 2\theta$

$= \frac{d}{dx} 2\theta$

$= 2\frac{d}{dx}\theta$

$= 2\frac{d}{dx}\cos^{-1} x$

$= 2 \times \frac{-1}{\sqrt{1-x^2}}$

$= \frac{d}{dx} \cos^{-1} \cos\theta$

$= \frac{d}{dx} \theta = \frac{d}{dx} \frac{1}{2}\cos^{-1} x = \frac{1}{2} \frac{(-1)}{\sqrt{1-x^2}}$

(9) $\frac{d}{dx} \cot^{-1} \frac{\sqrt{1-x^2}}{x}$

let $x = \sin\theta$

$\Rightarrow \frac{d}{dx} \cot^{-1} \frac{\sqrt{1-(\sin\theta)^2}}{\sin\theta} = \frac{d}{dx} \cot^{-1} \frac{\cos\theta}{\sin\theta}$

$= \frac{d}{dx} \cot^{-1} \cot\theta$

$= \frac{d}{dx} \theta = \frac{1}{\sqrt{1-x^2}} = \frac{d}{dx} \sin^{-1} x$

$= \frac{1}{\sqrt{1-x^2}}$

Q $\frac{d}{dx} \cos^{-1}(2t^2-1)$ = let $t = \cos\theta$

$\frac{d}{dt} \cos^{-1}(2\cos^2\theta - 1)$

$= \frac{d}{dt} \cos^{-1} \cos 2\theta$

$= \frac{d}{dt} 2\theta = 2\frac{d}{dt}\theta$

$= 2\frac{d}{dx}\cos^{-1} t$

$= 2 \times \frac{(-1)}{\sqrt{1-t^2}}$

Q $\frac{d}{dt} \cos^{-1}\left(\frac{1-t^2}{1+t^2}\right)$

let $t = \tan\theta$

~~$\frac{d}{dt} \cos^{-1} \left(\frac{1-\cos^2\theta}{1+\cos^2\theta}\right)$~~

$= \frac{d}{dt} \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \frac{d}{dt} \cos^{-1} \cos 2\theta = \frac{d}{dt} 2\theta$

$= 2\frac{d}{dt}\theta = 2\frac{d}{dt}\tan^{-1} t$

$= 2 \times \frac{1}{1+t^2}$

5. $\frac{d}{dx} \tan^{-1}\left(\frac{1}{1-x+x^2}\right)$

$= \frac{d}{dx} \tan^{-1}\left(\frac{1}{1-x(1-x)}\right) = \frac{d}{dx} \tan^{-1}\left[\frac{x+(1-x)}{1-x(1-x)}\right]$

Let $x = \tan\alpha$

$1-x = \tan\beta$

$= \frac{d}{dx} \tan^{-1}\left[\frac{\tan\alpha + \tan\beta}{1-\tan\alpha \cdot \tan\beta}\right] = \frac{d}{dx} \tan^{-1} \tan(\alpha+\beta)$

$= \frac{d}{dx} (\alpha+\beta)$

$= \frac{d}{dx} \tan^{-1}x + \tan^{-1}(1-x)$

$= \frac{1}{1+x^2} + \frac{1}{1+(1-x)^2}(-1)$

6. $\tan\theta = \left(\frac{1-t}{1+t}\right)$ find $\frac{d\theta}{dt} = ?$

$\Rightarrow \theta = \tan^{-1}\frac{1-t}{1+t}$ let $t = \tan z$

$\Rightarrow \theta = \tan^{-1} \frac{\tan\frac{\pi}{4} - \tan z}{1+\tan\frac{\pi}{4}\tan z}$

$\Rightarrow \theta = \tan^{-1}\tan\left(\frac{\pi}{4} - z\right)$

$\Rightarrow \theta = \frac{\pi}{4} - z$

7. $\frac{d}{dx} \sec\left(\frac{x^2+1}{x^2-1}\right)$

$= \frac{d}{dx} \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$

$= \frac{d}{dx} \cos^{-1}\left[-\left(\frac{1-x^2}{1+x^2}\right)\right]$

$\sin^{-1}(-x) = -\sin^{-1}x$

$\cos^{-1}(-x) = \pi - \cos^{-1}x$

$\tan^{-1}(-x) = -\tan^{-1}(x)$

$\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

$\sec^{-1}(-x) = \pi - \sec^{-1}x$

$= \frac{d}{dx} \pi - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$= \frac{d}{dx} \pi - \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \frac{d}{dx} \pi - \cos^{-1}\cos 2\theta$

$= \frac{d}{dx} \pi - 2\tan^{-1}x$

$= 0 - 2 \times \frac{1}{1+x^2}$

$\mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1}x$

$\mathrm{cosec}^{-1}(x) = \sin^{-1}\frac{1}{x}$

$\sec^{-1}x = \cos^{-1}\frac{1}{x}$

$\cot^{-1}x = \tan^{-1}\frac{1}{x}$

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

$\sec^{-1}x + \mathrm{cosec}^{-1}x = \frac{\pi}{2}$

8. $\tan^{-1}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$

$= \frac{d}{dx}\tan^{-1}\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}$

$= \frac{d}{dx}\tan^{-1}\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}$

$= \frac{d}{dx}\tan^{-1}\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}$

$= \frac{d}{dx}\tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) = \frac{d}{dx}\tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right]$

$= \frac{d}{dx}\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right)$

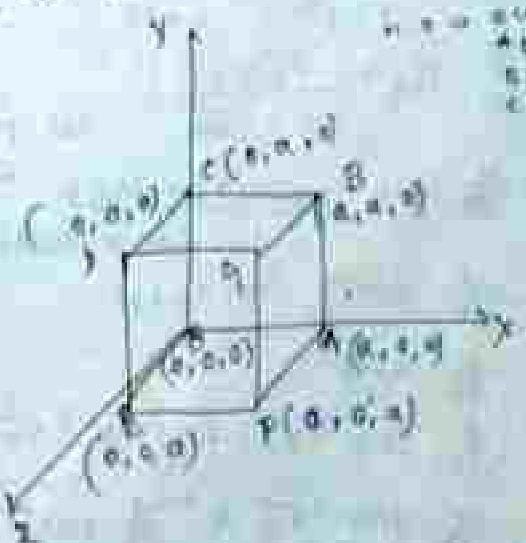
$= \frac{d}{dx}\frac{\pi}{4}+\theta = \frac{d}{dx}\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$

$\vec{OG} = a\hat{i}+a\hat{j}+a\hat{k}$

$\vec{AD} = -a\hat{i}+a\hat{j}+a\hat{k}$

$\cos\theta = \frac{\vec{OG}\cdot\vec{AD}}{|\vec{OG}||\vec{AD}|}$

$\Rightarrow \cos$



Differentiation by using logarithm.

Let $y = x^x$

By applying 'ln' in both sides, we have

$\Rightarrow \ln y = \ln x^x$

$\Rightarrow \ln y = x \ln x$

$\Rightarrow \frac{d}{dx}\ln y = \frac{d}{dx} x\ln x \Rightarrow \frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}\ln x + \ln x \frac{d}{dx}x$

$\Rightarrow \frac{dy}{dx} = y\left[x \times \frac{1}{x} + \ln x\right]$

$\Rightarrow \frac{dy}{dx} = x^x\left[1+\ln x\right]$

$\frac{d}{dx} x^{\sin x}$

Let $y = x^{\sin x}$

By applying

$\Rightarrow \ln y = \ln x^{\sin x}$

$\Rightarrow \ln y = \sin x \cdot \ln x$

$\Rightarrow \frac{d}{dx}\ln y = \frac{d}{dx}\sin x \cdot \ln x$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \sin x \frac{d}{dx}\ln x + \ln x \frac{d}{dx}\sin x$

$\Rightarrow \frac{dy}{dx} = y\left[\sin x \cdot \frac{1}{x} + \ln x \times \cos x\right]$

$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \ln x \cdot \cos x\right]$

$\frac{d}{dx}(\log x)^{\tan x}$

Let $y = (\log x)^{\tan x}$

By applying

$\Rightarrow \ln y = \ln(\log x)^{\tan x}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\tan x \ln(\log x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \tan x \frac{d}{dx}\ln(\log x) + \ln(\log x)\frac{d}{dx}\tan x$

$\Rightarrow \frac{dy}{dx} = y\left[\tan x \cdot \frac{1}{\log x}\cdot\frac{1}{x} + \ln(\log x)\cdot \sec^2 x\right]$

$\Rightarrow \frac{dy}{dx} = (\log x)^{\tan x}\left[\frac{\tan x}{x\log x} + \ln(\log x)\sec^2 x\right]$

$\frac{d}{dx} x^x + 5$

Let $y = x^x + 5$

By applying 'ln' in both sides, we have

$\Rightarrow \ln y = \ln(x^x + 5)$

$\Rightarrow \ln y = x\ln x + \ln 5$

$\Rightarrow \frac{d}{dx}\ln y = \frac{d}{dx}x\ln x + \frac{d}{dx}\ln 5$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}\ln x + \ln x\frac{d}{dx}x + 0$

$\Rightarrow \frac{dy}{dx} = y\left[x \times \frac{1}{x} + \ln x\right]$

~~$\Rightarrow \frac{dy}{dx} = y[1+\ln x]$~~ $\Rightarrow \frac{dy}{dx} = (x^x+5)\left[1+\ln x\right]$

$y = x^x + 5$

$\Rightarrow y = u + v, \Rightarrow u = x^x$

$\Rightarrow \ln u = \ln x^x \Rightarrow \ln u = x\ln x$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = x\frac{1}{x} + \ln x$

$\Rightarrow \frac{du}{dx} = u\left[1+\ln x\right] = x^x\left[1+\log x\right]$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$= x^x(1+\ln x)$

$y = x^{1/x} + (\sin x)^x$

$y = u + v \Rightarrow u = x^{1/x}$

$\Rightarrow \ln u = \ln x^{1/x}$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{1}{x} \times \frac{1}{x} + \ln x\frac{(-1)}{x^2}$

$\Rightarrow \frac{du}{dx} = x^{1/x}\left[\frac{1-\ln x}{x^2}\right]$ —— (1)

$v = (\sin x)^x$

$\Rightarrow \ln v = \ln(\sin x)^x$

$\Rightarrow \ln v = x\ln\sin x$

$\Rightarrow \frac{1}{v}\frac{dv}{dx} = x\frac{1}{\sin x}\cdot\cos x + \ln\sin x$

$\Rightarrow \frac{dv}{dx} = v\left[x\cot x + \ln\sin x\right]$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$= x^{1/x}\left[\frac{1-\ln x}{x^2}\right] + \left[x\cot x + \ln \sin x\right]$

Q. $(\cos x)^x + x^{\cos x}$

$\frac{d}{dx}(\cos x)^x + x^{\cos x}$

let $u = (\cos x)^x$

$\ln u = \ln(\cos x)^x$

$\Rightarrow \frac{d}{dx}\ln u = x\ln(\cos x) \Rightarrow \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\ln\cos x + \ln\cos x \frac{d}{dx}x$

$\Rightarrow \frac{du}{dx} = u\left[x \cdot \frac{1}{\cos x} \cdot -\sin x + \ln\cos x \cdot 1\right]$

$\Rightarrow \frac{du}{dx} = u\left[\frac{x}{\cos x} - x\tan x + \ln\cos x\right]$

let $v = x^{\cos x}$

let $v = \ln(x)^{\cos x}$

$\Rightarrow \frac{d}{dx}\ln v = \cos x \cdot \ln x$

$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \cos x \frac{d}{dx}\ln x + \ln x \frac{d}{dx}\cos x$

$\Rightarrow \frac{dv}{dx} = v\left[\cos x \cdot \frac{1}{x} + \ln x \cdot -\sin x\right]$

$\Rightarrow \frac{dv}{dx} = v\left[\frac{\cos x}{x} + \ln x \cdot -\sin x\right]$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = (\cos x)^x \left[-x\tan x + \ln\cos x\right] + x^{\cos x}\left[\frac{\cos x}{x} + \ln x \cdot -\sin x\right]$

Q. $y = x^{x^x}$

$\Rightarrow \ln y = \ln x^{x^x}$

$\Rightarrow \ln y = x^x \ln x$

$\Rightarrow \log(\ln y) = \log(x^x \ln x)$

$\Rightarrow \log(\ln y) = \log x^x + \log(\ln x)$

$\Rightarrow \log(\ln y) = x\ln x + \log(\ln x)$

$\Rightarrow \frac{1}{\ln y}\frac{d}{dx}\ln y = x\frac{d}{dx}\ln x + \ln x\frac{dx}{dx} + \frac{1}{\ln x}\frac{d}{dx}\ln x$

$\Rightarrow \frac{1}{\ln y} \times \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \ln x + \frac{1}{\ln x} \times \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = y\ln y\left[1 + \ln x + \frac{1}{x\ln x}\right]$

$\Rightarrow \frac{dy}{dx} = x^{x^x}\ln y\left[1 + \ln x + \frac{1}{x\ln x}\right]$

Q. $\frac{d}{dx}x^x$

$= \frac{d}{dx}e^{\ln x^x} = \frac{d}{dx}e^{x\ln x}$

$= e^{x\ln x}\frac{d}{dx}x\ln x$

$= x^x\left[x \times \frac{1}{x} + \ln x\right]$

$= x^x\left[1 + \ln x\right]$

Q. $\frac{d}{dx}x^{x^x}$

$= \frac{d}{dx}e^{\ln x^{x^x}} = \frac{d}{dx}e^{x^x \ln x}$

$= e^{x^x \ln x}\frac{d}{dx}x^x \ln x = x^{x^x}\left[x^x\frac{d}{dx}\ln x + \ln x\frac{d}{dx}x^x\right]$

$= x^{x^x}\left[x^x \frac{1}{x} + \ln x \frac{d}{dx}e^{\ln x^x}\right]$

$f(f) \cdot f(x)$

Derivative of Implicit Functions :-

Q. $x^2 + y^3 = 5$

$\Rightarrow \frac{d}{dx}(x^2 + y^3) = \frac{d}{dx}(5)$

$\Rightarrow 2x + 3y^2\frac{dy}{dx} = 0$

$\Rightarrow 3y^2\frac{dy}{dx} = -2x$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{3y^2}$

Q. $xy^2 + x^2y + 1 = 0$

$\Rightarrow \frac{d}{dx}xy^2 + x^2y + 1 = \frac{d}{dx}0$

~~$\Rightarrow \frac{d}{dx}xy^2 + \frac{d}{dx}x^2y + \frac{d}{dx}1 = 0$~~

~~$\Rightarrow x \cdot 2y\frac{dy}{dx} + 2x\frac{dy}{dx} = 0$~~

~~$\Rightarrow x \cdot 2y\frac{dy}{dx} = -2x$~~

~~$\Rightarrow \frac{dy}{dx} = \frac{-2x}{x \cdot 2y}$~~

$\Rightarrow x \cdot 2y\frac{dy}{dx} + y^2 + x^2\frac{dy}{dx} + y \cdot 2x = 0$

$\Rightarrow x \cdot 2y\frac{dy}{dx} + x^2\frac{dy}{dx} = -y^2 - 2xy$

$\Rightarrow \frac{dy}{dx}(2xy + x^2) = -y^2 - 2xy$

$\Rightarrow \frac{dy}{dx} = -y^2 - 2xy / 2xy + x^2$

Q. $x^2 + 3y^2 = 5$

$\frac{d}{dx} x^2 + 3y^2 = \frac{d}{dx} 5$

$\Rightarrow 2x + 3 \cdot 2y \frac{dy}{dx} = 0$

$\Rightarrow 3 \cdot 2y \frac{dy}{dx} = -2x$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{6y} = \frac{-x}{3y}$

Q. $\frac{dy}{dx}\ x^2 + 3y^2 - 5 = 0$

$\Rightarrow \frac{dy}{dx} = \left[\frac{-2x}{3 \cdot 2y}\right]$

$\Rightarrow \frac{dy}{dx} \left[\frac{-2x}{6y}\right]$

Q. $y^x = x^{\sin y}$

$\ln y^x = \ln x^{\sin y}$

$\Rightarrow x \ln y = \sin y \ln x$

$\Rightarrow x \cdot \frac{d}{dx} \ln y + \ln y \frac{d}{dx} x = \sin y \frac{d}{dx} \ln x + \ln x \frac{d}{dx} \sin y$

$\Rightarrow x \cdot \frac{1}{y}\frac{dy}{dx} + \ln y = \sin y \times \frac{1}{x}\frac{dy}{dx} + \ln x \cdot \cos y \frac{dy}{dx}$

$\Rightarrow \frac{x}{y} \frac{dy}{dx} + \ln y = \frac{\sin y}{x} \frac{dy}{dx} + \ln x \cdot \cos y$

$\Rightarrow \frac{dy}{dx} \left(\frac{x}{y} - \ln x \cdot \cos y\right] = \frac{\sin y - x \ln y}{x}$

$\Rightarrow \frac{dy}{dx} \left[\frac{x}{y} - \ln x \cdot \cos y\right] = \frac{\sin y - x \ln y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y\,[\sin y - x \ln y]}{x - y \ln x \cos y}$

Imp

Q-10 $\sin(x+y) = y \cos(x+y)$

$\Rightarrow y = \tan(x+y) \Rightarrow x + y = \tan^{-1} y$

$\Rightarrow \frac{d}{dx}(x+y) = \frac{d}{dx} \tan^{-1} y \Rightarrow 1 + \frac{dy}{dx} = \frac{1}{1+y^2} \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} - \frac{1}{1+y^2} \frac{dy}{dx} = -1$

$\Rightarrow \frac{dy}{dx} \left[1 - \frac{1}{1+y^2}\right] = -1 \Rightarrow \frac{dy}{dx} \left[\frac{1+y^2-1}{1+y^2}\right] = -1$

$\Rightarrow \frac{dy}{dx} = \frac{-(1+y^2)}{y^2}$

Differentiation of parametric functions :-

$\frac{dx}{dt} = 2a$ —— (i) $\Rightarrow \frac{dy}{dt} = \frac{1}{2} a \times 2t = at$ —— (ii)

$\frac{dy/dt}{dx/dt} = \frac{\frac{1}{2}a\,2t}{2a} = \frac{at}{2a} = \frac{t}{2}$

Q. $x = a\cos\theta$, $y = a\sin\theta$

$\frac{dx}{d\theta} = a(-\sin\theta)$, $\frac{dy}{d\theta} = a\cos\theta$

$\Rightarrow \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \cos\theta}{-a\sin\theta} \Rightarrow \frac{dy}{dx} = -\cot\theta$

Q. $x = a\cos^3 t$; $y = a\sin^3 t$, at $= \frac{\pi}{4}$

$\frac{dx}{dt} = -a\,3\cos^2 t \cdot \sin t$ —— (i)

$\frac{dy}{dt} = a\,3\sin^2 t \cdot \cos t$ —— (ii)

$\frac{dy}{dt} / \frac{dx}{dt} = \frac{a\,3\sin^2 t \cdot \cos t}{-a\,3\cos^2 t \cdot \sin t} \Rightarrow \frac{dy}{dx} = -\tan t$

$\Rightarrow \frac{dy}{dx} = -\tan\frac{\pi}{4} \Rightarrow \frac{dy}{dx} = -1$

Derivative of a function w.r.t another function :-

$y = \sin x$ $z = \log x$

Q. $y = (\sqrt{x})^{x^{x^{\cdots \infty}}}$ $\Rightarrow y = (\sqrt{x})^{y}$

$\Rightarrow y = (x^{1/2})^{x^{x^{\cdots \infty}}}$

$\Rightarrow y = (x^{1/2})^{x^{x^{\cdots \infty}}}$ $\frac{1}{2}$

$\Rightarrow y = (x^{x^{\cdots \infty}})$

$\Rightarrow y^2 = x^{x^{\cdots \infty}}$

$\Rightarrow y^2 = x^{y^2} \Rightarrow 2\ln y = y^2 \ln x$

$\Rightarrow 2\frac{d}{dx}\ln y = \frac{d}{dx}\ln y + \ln y \frac{d}{dx} x$

$\Rightarrow 2\frac{1}{y}\frac{dy}{dx} = \frac{x}{y}\frac{dy}{dx} \ln y$

$\Rightarrow \frac{2}{y}\frac{dy}{dx} = \frac{x}{y}\frac{dy}{dx} + \ln y$

$\Rightarrow \frac{dy}{dx}\left[\frac{2}{y} - \frac{x}{y}\right] = \ln y$ $\Rightarrow \frac{dy}{dx}\left[\frac{2-x}{y}\right] = \ln y$

$\Rightarrow \frac{dy}{dx} \Rightarrow 2\ln y = y^2 \ln x$ $\Rightarrow 2 \times \frac{1}{y}\frac{dy}{dx} = y^2 \frac{1}{x} + (\ln x) 2y \frac{dy}{dx}$

$\Rightarrow \frac{2}{y}\frac{dy}{dx} = \frac{y^2}{x} + (\ln x) 2y \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}\left(\frac{2}{y} - (\ln x) 2y\right) = \frac{y^2}{x}$ $\Rightarrow \frac{dy}{dx} \cdot \left[\frac{2 - \ln x\, 2y^2}{y}\right] = \frac{y^2}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{\frac{y^2}{x}}{\frac{2 - \ln x\, 2y^2}{y}}$ $\Rightarrow \frac{dy}{dx} = \frac{y^2}{x} \times \frac{y}{2 - \ln x\, 2y^2} = \frac{y^3}{x[2 - \ln x\, 2y^2]}$

$\Rightarrow \ln y = y \ln\sqrt{x}$

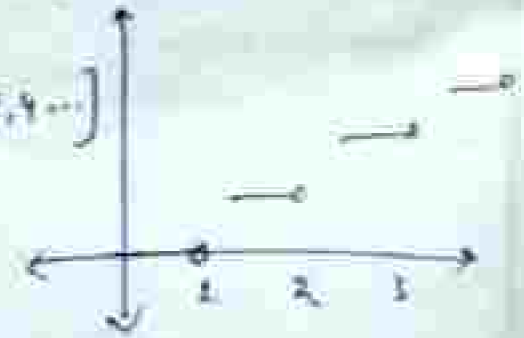
$\Rightarrow \frac{1}{y}\frac{dy}{dx} = y \cdot \frac{d}{dx}\ln\sqrt{x} + \ln\sqrt{x}\frac{dy}{dx}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = y \cdot \frac{1}{x} + \ln\sqrt{x}\frac{dy}{dx}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{y}{x} + \ln\sqrt{x}$

Q. Given an example of a function which is not continuous as well as not differentiable at a point.

• $\frac{d}{dx}[x]$ at $(1, 2, 3, 4 \ldots)$



$(x+1)^{[x]}$

$x(x+1)$

$x \cdot \frac{d}{dx} x+1 + x+1 \frac{d}{dx} x$

$x + x + 1 = 2x + 1$

$\frac{d}{dx} e^{[x] \ln(x+1)}$ when $3 \le x < 4$

$\Rightarrow \frac{d}{dx} e^{3 \ln(x+1)}$ $\Rightarrow \frac{d}{dx}(x+1)^3$ $\Rightarrow 3(x+1)^2$

Q. Prove that the derivative of a non constant even function is a odd function? **Imp.**

Even

$f(x) = f(-x)$

$\Rightarrow f'(-x)\frac{d}{dx}(-x) = f'(x)$

$\Rightarrow f'(-x)(-1) = f'(x)$

$\Rightarrow f'(-x) = -f'(x)$

Prove that the derivative of a non constant odd function is a even function

$f(-x) = -f(x)$

$\Rightarrow f'(-x)\frac{d}{dx}(-x) = -f'(x)$

$\Rightarrow f'(-x)(-1) = -f'(x)$

$\Rightarrow f'(-x) = f'(x)$ (proved)

Second Derivative :-

$y = x^2 \Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}x^2$

$\Rightarrow \frac{dy}{dx} = 2x$ (first derivative) $y_1 = Dy = f'x$

$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}2x$

$\Rightarrow \frac{d^2y}{dx^2} = 2$ —— (Second derivative)

$y_2 = D^2y = f''(x)$

$\Rightarrow \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$

Imp. $\Rightarrow \frac{d^3y}{dx^3} = 0$ (Third Derivative)

Q. $y = \tan^{-1} x$, then P.T $(1+x^2)y_2 + 2xy_1 = 0$

$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}\tan^{-1} x$

$x \to 0$ ...

(Q-1, 2, 3, 4)

Q. If $y = e^{a\sin^{-1}x}$ then P.T.

$\Rightarrow \frac{d}{dx} y = \frac{d}{dx} e^{a\sin^{-1}x}$

$\Rightarrow y_1 = e^{a\sin^{-1}x} \frac{d}{dx} a\sin^{-1}x$

$\Rightarrow y_1 = y\, a \frac{1}{\sqrt{1-x^2}}$ $\Rightarrow y_1 = \frac{ya}{\sqrt{1-x^2}}$

$\Rightarrow y_1 \sqrt{1-x^2} = ya$

$\Rightarrow (1-x^2) y_1^2 = y^2 a^2$

$\Rightarrow 1-x^2 \frac{d}{dx} y_1^2 + y_1^2 \frac{d}{dx} 1-x^2 = a^2 2y \frac{dy}{dx}$

$\Rightarrow 1-x^2\, 2y_2 + y_1^2 \cdot 2x = a^2 2y y_1$

__Important facts:-__

- If f & g are two continuous function then $f+g$, $f-g$, kf, fg, $\frac{1}{g}$ $(g \neq 0)$, $\frac{1}{f}$ $(f \neq 0)$ are continuous function.
- Polynomial function is continuous everywhere.
  eg. $f(x) = 2x - 3$
  $f(x) = 5x^2 - 2x + 7$
  $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$
- Rational function are continuous in their respective domain.
  eg. $\frac{1}{x}$
- Modulus function is continuous everywhere.
- $|f|$ is continuous if f is continuous
- Logarithmic function are continuous in their respective domain.

- __Exponential function:-__
- e^x is continuous everywhere.
- Sine and cos function continuous everywhere.
- Tan, cot, sec & cosec are continuous in respective domain.
- If f is a continuous function defined on $[a, b]$ such that $f(a) f(b) < 0$ then there exist at least 1 solution of the equation $f(x) = 0$ in the open interval (a, b)

Q. Prove that $x = \cos x$ (for $\exists$ some) $x \in (0, \frac{\pi}{2})$

Sol^n $\Rightarrow x - \cos x = 0$
$f(x) = x - \cos x$
Since x is continuous because x is polynomial function
$\cos x$ = also continuous everywhere
$x - \cos x$ is continuous everywhere
$f(a) = f(0) = 0 - \cos 0 = -1$
$f(b) = f(\frac{\pi}{2}) = \frac{\pi}{2} - \cos\frac{\pi}{2}$
$= \frac{\pi}{2}$
$f(a) \cdot f(b) = (-1)\frac{\pi}{2}$
$f(x) = 0 \Rightarrow x - \cos x = 0$
$\Rightarrow x = \cos x$ (proved)

- If 'f' is continuous function $[a, b]$ and k is any real number between $f(a)$ & $f(b)$ then there exist atleast one solution of the equation $f(x) = k$ in the open interval.

Q. Prove that the expression $2^x + x^2$ attains the value for some value of $x \in (0, 1)$
$f(x) = 2^x + x^2$

$f(0) = 2^0 + 0^2 = 1$

$f(1) = 2^1 + 1^2 = 3$

$f(0) < 2 < f(1)$

$\Rightarrow f(x) = 2$

If two function a, b

(i) continuous on the closed interval [a, b]

(ii) Differentiable on the open interval (a, b)

(iii) $f(a) = f(b)$

then there exists at least a point $c \in (a, b)$

$f'(c) = 0$

Q. What is the geometrical significance of Lagrange mean value theorem?

Under the hypothesis of Lagrange mean value theorem the tangent drawn to the curve $y = f(x)$ at $x = c$ will be parallel to the chord joining $\{a, f(a)\}$ an $\{b, f(b)\}$

Q. Verify Lagrange mean value theorem $f(x) = 2x^2 - 3x + 7$ on $[1, 2]$

Sol. $\because f(x) = 2x^2 - 3x + 7$ is a polynomial function, hence

(i) it is continuous in $[1, 2]$

(ii) it is diff. in $(1, 2)$

$f(a) = f(1) = 2(1)^2 - 3(1) + 7 = 6$

$f(b) = f(2) = 2(2)^2 - 3(2) + 7 = 9$

$f'(x) = 4x - 3$

By LMV

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow 4c - 3 = \frac{9 - 6}{2 - 1}$$

$\Rightarrow 4c - 3 = 3 \Rightarrow 4c = 6 \Rightarrow c = \frac{6}{4} = \frac{3}{2} = 1.5$

$1.5 \in (1, 2)$

Cauchy's Mean Value Theorem

If f & g are two functions

(i) both are cont. in the closed interval $[a, b]$

(ii) both are diff. in the open interval (a, b)

(iii) $g'(c) \neq 0$

then there exist at least a point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Q. Prove by using Lagrange mean value theorem $\sin x < x$, $\exists x \in (0, \frac{\pi}{2})$

Sol. $\therefore$ ~~sin~~ function $\sin x < x$

$\Rightarrow \sin x - x < 0$

$f(x) = \sin x - x$

$\therefore \sin x$ is cont. everywhere and x is cont. everywhere

$\Rightarrow f(x) = \sin x - x$ is cont. $[0, \frac{\pi}{2}]$

$\therefore \sin x$ is diff. everywhere and x is diff. everywhere

$\Rightarrow f(x) = \sin x - x$ is diff $[0, \frac{\pi}{2}]$

$a = 0, \quad b = x \in (0, \frac{\pi}{2})$

$f(a) = \sin 0 - 0 = 0$

$f(b) = \sin x - x$

$f'(x) = \cos x - 1$

By LMV

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \cos c - 1 = \frac{\sin x - x - 0}{x - 0}$$

$$\Rightarrow \cos c - 1 = \frac{\sin x - x}{x}$$

$\Rightarrow \frac{\sin x - x}{x} < 0 \quad \because \cos c - 1 < 0$

$\Rightarrow \sin x - x < 0$

$\Rightarrow \sin x < x$ (proved)

Differentiability :-

If f is a real valued function defined on $[a, b]$, then 'f' is differentiable at $x = c \in (a, b)$ if and only if

$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists finitely

LHL

$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$

$= \lim_{h \to 0} \frac{f(c-h) - f(c)}{c - h - c}$

$= \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$

RHL

$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$

$= \lim_{h \to 0} \frac{f(c+h) - f(c)}{c + h - c}$

$= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

LHD $f'(c^-) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$

RHD $f'(c^+) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

Q

$f(c) = f(0) = |0| = 0$

$f(c-h) = f(0-h) = |0-h| = h$

$f(c+h) = f(0+h) = |0+h| = h$

$f'(c^-) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$

$= \lim_{h \to 0} \frac{h - 0}{-h} = -1$

$f'(c^+) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

$= \lim_{h \to 0} \frac{h - 0}{h} = 1$

$\therefore f'(c^-) \neq f'(c^+)$

$\Rightarrow f$ is not differentiable at $x = 0$.

Q $f(x) = \left|1 - \frac{1}{x}\right|$ at $x = 1$

$f(c) = f(1) = \left|1 - \frac{1}{1}\right| = |0| = 0$

$f(c-h) = f(1-h) = \left|1 - \frac{1}{1-h}\right| = \left|\frac{1-h-1}{1-h}\right| = \frac{h}{1-h}$

$f(c+h) = f(1+h) = \left|1 + \frac{1}{1+h}\right| = \left|\frac{1+h-1}{1+h}\right| = \frac{h}{1+h}$

$f'(c^-) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$

$= \lim_{h \to 0} \frac{\frac{h}{1-h} - 0}{-h} = \lim_{h \to 0} \frac{\frac{h}{1-h}}{-h}$

$= \lim_{h \to 0} \frac{h}{1-h} \times \frac{1}{-h} = -1$

RHD $f'(c^+) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

$= \lim_{h \to 0} \frac{\frac{h}{1+h} - 0}{h}$

$= \lim_{h \to 0} \frac{h}{1+h} \times \frac{1}{h} = 1$

Q $f(x) = x^2 |x|$ at $x = 0$

$f(c) = 0^2 |0| = 0$

$f(c+h) = (0+h)^2 |0+h| = h^3$

$f(c-h) = (0-h)^2 |0-h| = -h^3$

$f'(c^-) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = \frac{-h^3 - 0}{-h} = h^2$

$f'(c^+) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \frac{h^3 - 0}{h} = h^2$

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Integration

The process by which we can get anti derivative of a function.

Integration is the reverse process of differentiation.

Antiderivatives also called primitive.

Integration is a one-many relation i.e. a function can have more than one antiderivatives.

A constant is added after every integration in order to represent many anti derivatives.

$$\int f(x)\,dx = F(x) + c$$

$f(x)$ ↓ Integrand

$F(x)$ ↓ Integral

Formulae :-

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad , n \neq -1$$

$$\int x^{2/3}\,dx = \frac{x^{5/3}}{5/3} + c = \frac{3}{5} x^{5/3} + c$$

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\cdot dx = \frac{x^{1/2}}{1/2} + c$$

$$= \boxed{2\sqrt{x} + c}$$

$$\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + c$$

Q-1 $$\int x^{0.8}\,dx = \frac{x^{0.8+1}}{0.8+1} + c$$

$$= \frac{x^{1.8}}{1.8} + c$$

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx$$

$$= \frac{x^{-1}}{-1} + c = -x^{-1} + c$$

$$\int x^{-1}\,dx = \frac{x^0}{0} + c$$

$$\boxed{\int \frac{1}{x^2}\,dx = \frac{-1}{x} + c}$$

$$\boxed{\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c}$$

$$\boxed{\int a^x\,dx = \frac{a^x}{\ln a} + c, \; a > 0, \; a \neq 1}$$

$$\int e^x\,dx = e^x + c$$

Trignometry formulae :-

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$$

$$\int \sec x \tan x\,dx = \sec x + c$$

$$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + c$$

Q. $\int \left(\frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right) dx$

$= \int \frac{x^2 + (1-x^2)}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-x^2}} dx$

$= \sin^{-1}x + C$

Q. $\int \frac{\cos 2x}{\cos x - \sin x} dx$

$= \int \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x} dx$

$= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x - \sin x)} dx$

$= \int \cos x + \sin x \, dx = \sin x - \cos x + C$

Integration by substitution :-

Q. $\int \cos 7x \, dx$ Let $t = 7x$, $\frac{dt}{dx} = 7 \Rightarrow dx = \frac{dt}{7}$

$\Rightarrow \int \cos t \frac{dt}{7}$

$= \frac{1}{7} \int \cos t \, dt = \frac{1}{7} \sin t + C$

$= \frac{1}{7} \sin 7x + C$

Q. $\int \frac{dx}{x[1+(\ln x)^2]}$ Let $t = \ln x$, $\frac{dt}{dx} = \frac{1}{x}$, $\Rightarrow dx = \frac{dt}{x}$

$= \int \frac{dt}{1+t^2}$

$= \tan^{-1} t + C$

$= \tan^{-1}(\ln x) + C$

Q. $\int \sec e^x \cdot \tan e^x \cdot e^x dx$ Let $e^x = t$, $\Rightarrow \frac{dt}{dx} = e^x$, $\Rightarrow dt = e^x dx$

$= \int \sec t \cdot \tan t \cdot dt$

$= \sec t + C$

$= \sec(e^x) + C$

Q. $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} dx$ Let $t = \sqrt{x}$, $\Rightarrow \frac{dt}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dx = 2\sqrt{x}\,dt$

$= \int \sec^2 t \cdot 2\,dt$

$= 2\tan t + C = 2\tan\sqrt{x} + C$

Q. $\int \frac{\operatorname{cosec}^2 x}{11 + \cot x} dx$ Let $t = 11 + \cot x$, $\frac{dt}{dx} = -\operatorname{cosec}^2 x$, $\Rightarrow dt = -\operatorname{cosec}^2 x \, dx$

$\Rightarrow -\int \frac{dt}{t}$

$= -\ln|11 + \cot x| + C$

Shortcut

$\int f(x)\,dx = F(x) + C$

$\int f(\lambda x)\,dx = \frac{F(\lambda x)}{\lambda} + C$

Q. $\int \cos \frac{3x}{2} dx = \frac{\sin \frac{3x}{2}}{3/2} + C = \frac{2}{3} \sin \frac{3x}{2} + C$

Q. $\int \sec \sqrt{3}x \tan \sqrt{3}x \, dx = \frac{\sec \sqrt{3}x}{\sqrt{3}} + C$

Q. $\int a^{5x} dx = \frac{a^{5x}}{5(\ln a)} + C$

Q. $\int \operatorname{cosec}(x+2) \cot(x+2)\, dx$ Let $t = x+2$, $\frac{dt}{dx} = 1$, $\Rightarrow dt = dx$

$= \int \operatorname{cosec} t \cot t \, dt$

$= -\operatorname{cosec}(x+2) + C$

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$\int \sin(6x-5)\,dx$ $\quad$ let $t = 6x-5$, $\frac{dt}{dx} = 6 \Rightarrow dx = \frac{dt}{6}$

$= \int \sin t \,\frac{dt}{6}$

$= \frac{1}{6}\int \sin t \,dt = -\frac{\cos(6x-5)}{6} + C$

$\int \frac{1}{17x+19}\,dx = \frac{\ln|17x+19|}{17} + C$

$\int \frac{dx}{\sqrt{5x-2}}$ $\quad$ $t = 5x-2$, $\frac{dt}{dx} = 5 \Rightarrow dt = 5\,dx$

$= \int \frac{5\,dt}{\sqrt{t}}$ $\quad$ $= 5\int t^{-1/2} = 5\,\frac{t^{1/2}}{1/2} + C = \frac{10}{5}(5x-2)^{1/2} + C$

$= \frac{2\sqrt{5x-2}}{5} + C$

Formula derive :-

$\int \tan x\,dx$

$= \int \frac{\sin x}{\cos x}\,dx$ $\quad$ set $t = \cos x$, $\frac{dt}{dx} = -\sin x \Rightarrow -dt = \sin x\,dx$

$= \int \frac{-dt}{t}$

$= -\ln|t| + C$

$= -\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + C$

$= \boxed{\ln|\sec x| + C}$

$\int \cot x\,dx$

$= \int \frac{\cos x}{\sin x}\,dx$ $\quad$ $t = \sin x \Rightarrow \frac{dt}{dx} = \cos x \Rightarrow dt = \cos x\,dx$

$= \int \frac{dt}{t} = \ln|t| + C$

$= \ln|\sin x| + C$

$\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\,dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$ $\quad$ $t = \sec x + \tan x$, $\frac{dt}{dx} = \sec x \tan x + \sec^2 x$

$= \int \frac{dt}{t}$

$= \ln|t| + C = \ln|\sec x + \tan x| + C$

$\int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)}$

$= \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cdot \cot x}{(\operatorname{cosec} x - \cot x)}$ $\quad$ $t = \operatorname{cosec} x - \cot x$, $\frac{dt}{dx} = -\operatorname{cosec} x \cdot \cot x + \operatorname{cosec}^2 x$, $\Rightarrow dt = \operatorname{cosec}^2 x - \operatorname{cosec} x \cot x$

$= \int \frac{dt}{t}$

$= \ln|t| + C = \ln|\operatorname{cosec} x - \cot x| + C$

$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$

$\int (\sec 2x - 3)^2\,dx$

$= \int \sec^2 2x + 9 - 2 \cdot \sec 2x \cdot 3\,dx$

$= \int \sec^2 2x\,dx + \int 9\,dx - 6\int \sec 2x\,dx$

$= \frac{\tan 2x}{2} + 9x - 6\ln\left|\frac{\sec 2x}{2} + \frac{\tan 2x}{2}\right| + C$

$= \frac{1}{2}\left[\tan 2x - 3\ln|\sec 2x + \tan 2x|\right] + 9x + C$

$\int 3^x e^{x}\,dx$

$= \int (3e)^x\,dx = \frac{(3e)^x}{\ln(3e)} + C = \frac{3^x e^x}{\ln 3 + 1} + C$

Any (integral) power of $\tan x$ & $\cot x$:-

$\int (\tan x)^n dx = \int (\tan x)^{n-2} \tan^2 x \, dx$

$= \int (\tan x)^{n-2} (\sec^2 x - 1) dx$

$\int (\cot x)^n dx = \int (\cot x)^{n-2} \cot^2 x \, dx$

$= \int (\cot x)^{n-2} (\operatorname{cosec}^2 x - 1) dx$

Q. $\int \tan^3 x \, dx = \int \tan x \cdot \tan^2 x \, dx$

$= \int \tan x (\sec^2 x - 1) dx = \int \tan x \sec^2 x \, dx - \int \tan x \, dx$

$= \int t \, dt - \int \tan x \, dx$ let $t = \tan x$, $dt = \sec^2 x \, dx$

$= \frac{t^2}{2} - \ln|\sec x| + c$

Q. $\int \cot^6 x \, dx$

$= \int \cot^4 x \cdot \cot^2 x \, dx = \int \cot^4 x \cdot (\operatorname{cosec}^2 x - 1) dx$

$= \int \cot^4 x \operatorname{cosec}^2 x \, dx - \int \cot^4 x \, dx$ let $t = \cot x$

$= \int t^4 (-dt) - \int \cot^2 x (\operatorname{cosec}^2 x - 1) dx$ $\Rightarrow \frac{dt}{dx} = -\operatorname{cosec}^2 x$

$= -\int t^4 dt - \int \cot^2 x \operatorname{cosec}^2 x - \cot^2 x \, dx$ $\Rightarrow -dt = \operatorname{cosec}^2 x \, dx$

$= -\int t^4 dt - \int t(-dt) + \int \cot^2 x \, dx$

$= -\int t^4 dt + \int t^2 dt + \int \operatorname{cosec}^2 x - 1 \, dx$

$= -\frac{t^5}{5} + \frac{t^3}{3} + (-\cot x - x) + C$

Q. $\int \tan^5 x \, dx$

$= \int \tan^3 x \tan^2 x \, dx$

$= \int \tan^3 x (\sec^2 x - 1) dx$

$\int \sout{\sec^{70} \tan x \; dx}$

Exercise - 9(d)

formulae :-

(1) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$

$\Rightarrow \int \frac{dx}{a^2 - x^2}$ Let $x = a\sin\theta \Rightarrow \frac{dx}{d\theta} = a\cos\theta$

$\Rightarrow dx = a\cos\theta \, d\theta$

$= \int \frac{a\cos\theta \, d\theta}{\sqrt{a^2 - a^2\sin^2\theta}} = \int \frac{a\cos\theta \, d\theta}{\sqrt{a^2(1-\cos^2\theta)}}$

$= \int \frac{a\cos\theta \, d\theta}{a\cos\theta} = \theta + c$

$= \sin^{-1}\frac{x}{a} + c$

$\int \frac{dx}{\sqrt{8 - x^2}} = \int \frac{dx}{\sqrt{(2\sqrt{2})^2 - x^2}}$

$= \sin^{-1}\frac{x}{2\sqrt{2}} + c$

$\int \frac{dx}{\sqrt{11 - 4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{(\sqrt{11})^2 - (2x)^2}}$

$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{11}}\right) + c$

or $\int \frac{dx}{\sqrt{4\left(\frac{11}{4} - x^2\right)}} = \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{\sqrt{11}}{2}\right)^2 - x^2}} = \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{11}}\right) + c$

$\int \frac{dx}{\sqrt{7 - 3x^2}} = \int \frac{dx}{\sqrt{(\sqrt{7})^2 - ((\sqrt{3})x)^2}}$

$= \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{\sqrt{3}x}{\sqrt{7}}\right) + c$

$\int \frac{dx}{x\sqrt{17 - (\ln x)^2}}$ $\quad z = \ln x$

$\Rightarrow \frac{dz}{dx} = \frac{1}{x} \Rightarrow dz = \frac{dx}{x}$

$$= \int \frac{dx}{\sqrt{(\sqrt{17})^2 - (x)^2}} = \sin^{-1}\left(\frac{x}{\sqrt{17}}\right) + c$$

$$= \sin^{-1}\left(\frac{\ln x}{\sqrt{17}}\right) + c$$

$$\int \frac{dx}{\sqrt{9 - x^2 - 4x}} = \int \frac{dx}{\sqrt{-x^2 - 4x + 9}}$$

$$= \int \frac{dx}{\sqrt{9 - (x^2 + 4x + 4 - 4)}} = \int \frac{dx}{\sqrt{9 - (x+2)^2 + 4}}$$

$$= \int \frac{dx}{\sqrt{13 - (x+2)^2}} = \int \qquad t = x + 2, \quad dt = dx$$

$$= \int \frac{dt}{\sqrt{(\sqrt{13})^2 - (t)^2}} = \sin^{-1}\left(\frac{t}{\sqrt{13}}\right) + c$$

$$= \sin^{-1} \frac{x+2}{\sqrt{13}} + c$$

Form - 02

$$\boxed{\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c}$$

$t = a\tan\theta$
$\Rightarrow dx = a\sec^2\theta$

$$= \int \frac{a\sec^2\theta \, d\theta}{a^2 + a^2\tan^2\theta}$$

$$= \int \frac{a\sec^2\theta \, d\theta}{a^2 \sec^2\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{dx}{2 + x^2} = \int \frac{dx}{(\sqrt{2})^2 + x^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$$

$$\int \frac{dx}{3 + 7x^2} = \int \frac{dx}{(\sqrt{3})^2 + (\sqrt{7}x)^2} = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{7}} \tan^{-1}\left(\frac{\sqrt{7}x}{\sqrt{3}}\right) + c$$

$$= \frac{1}{\sqrt{21}} \tan^{-1}\left(\frac{\sqrt{7}x}{\sqrt{3}}\right) + c$$

$$\int \frac{e^x \, dx}{25 + e^{2x}} = \int \frac{e^x \, dx}{(5)^2 + (e^x)^2} \qquad t = e^x, \quad dt = e^x dx$$

$$= \int \frac{dt}{(5)^2 + (t)^2} = \tan^{-1}\left(\frac{t}{5}\right) + c$$

$$= \frac{1}{5}\tan^{-1}\left(\frac{e^x}{5}\right) + c$$

Form - 03

$$\boxed{\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c}$$

Let $x = a\sec\theta$

$\Rightarrow \frac{dx}{d\theta} = a\sec\theta\tan\theta \Rightarrow dx = a\sec\theta\tan\theta \, d\theta$

$$\int \frac{a\sec\theta\tan\theta \, d\theta}{a\sec\theta\sqrt{a^2\sec^2\theta - a^2}} = \int \frac{\tan\theta \, d\theta}{\sqrt{a^2\tan^2\theta}}$$

$$= \int \frac{\tan\theta \, d\theta}{a\tan\theta} = \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$$

Form - 04

$$\boxed{\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left|x + \sqrt{x^2 + a^2}\right| + c}$$

Let $x = a\tan\theta$
$\frac{dx}{d\theta} = a\sec^2\theta$
$\Rightarrow dx = a\sec^2\theta \, d\theta$

$$\int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$= \int \frac{a\sec^2\theta \, d\theta}{\sqrt{a^2\tan^2\theta + a^2}}$$

$$= \int \frac{a\sec^2\theta \, d\theta}{\sqrt{a^2\sec^2\theta}} = \int \frac{a\sec^2\theta \, d\theta}{a\sec\theta}$$

$$= \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + c$$

Integration by parts :-

I - Inverse trigonometry - $\sin^{-1}x$

L - Logarithm - $\ln$, $\log_3 x$

A - Algebraic - $\sqrt{x}$, x

T - Trigonometric - sec cosec

E - Exponential - e^x, e^{2x}, 2^x, $\frac{a^x}{b^x}$

$\int (u \cdot v)\, dx$

$$= u\int v\,dx - \int\left(\frac{d}{dx}u \int v\,dx\right)dx$$

$\int x e^x$

$= x\int e^x dx - \int\left(\frac{d}{dx}x \int e^x dx\right)dx$

$= x e^x - \int e^x dx$

$= x e^x - e^x + C$

$\int x \cos x\, dx$

$= x\int \cos x\, dx - \int\left(\frac{d}{dx}x \int \cos x\, dx\right)dx$

$= x\sin x - \int 1 \sin x\, dx$

$= x\sin x + \cos x + C$

$\int x^2 e^x dx$

$= x^2\int e^x dx - \int\left(\frac{d}{dx}x^2 \int e^x dx\right)dx$

$= x^2 e^x - \int 2x \cdot e^x dx$

$= x^2 e^x - \left[2x\int e^x dx - \int\left(\frac{d}{dx}2x \int e^x dx\right)dx\right]$

$= x^2 e^x - \left[2x e^x - 2\int e^x dx\right]$

$= x^2 e^x - 2x e^x + 2e^x + C$

$A+B+C = ABC = \sqrt{\frac{xr}{yz}} + \sqrt{\frac{yr}{zr}} + \sqrt{\frac{zr}{xy}} - \sqrt{\frac{r^3 xyz}{xyz \cdot xyz}}$

$= \frac{x\sqrt{r}}{xyz} + \frac{y\sqrt{r}}{\sqrt{xyz}} + \frac{z\sqrt{r}}{\sqrt{xyz}} - \frac{r\sqrt{r}}{\sqrt{xyz}}$

$= \frac{\sqrt{r}}{xyz}(x+y+z - r)$

$= \frac{\sqrt{r}}{xyz}(r - r) = 0$

from LHS

$\tan^{-1}\left(\frac{0}{1-AB-BC-AC}\right)$

$= 0 + \pi$

$= \pi$

Partial fraction Method :-

partial fraction

The method by which a bigger partial fraction can be broken into smaller parts is called partial fraction. (Highest p.f method is only aplicable for p.f (Highest power of Nr < Degree of Dr)

form : 01 (Linear factor)

[Every linear factor takes single constant]

$\int \frac{2x}{(x-3)(x-2)}$

$\int \frac{3x+1}{(x-1)(x-3)} dx$

Let $\frac{3x+1}{(x-1)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-3)}$

$\frac{3x+1}{(x-1)(x-3)} = \frac{A(x-3) + B(x-1)}{(x-1)(x-3)}$

$\Rightarrow 3x+1 = A(x-3) + B(x-1)$

when $x = 3$

$\Rightarrow 9+1 = B \times 2 \Rightarrow B = 5$

when $x = 1$

$\Rightarrow 3+1 = A - 2A \Rightarrow -2A = 4$

$\Rightarrow A = -2$

So $\int \frac{3x+1}{(x-1)(x-3)} dx = \int \frac{-2}{x-1} dx + \int \frac{5}{x-3} dx$

$= -2\ln|x-1| + 5\ln|x-3| + c$

$\int \frac{x^2-3}{(2x-1)(x+3)(x+1)} dx$

$\Rightarrow \frac{x^2-3}{(2x-1)(x+3)(x+1)} = \frac{A}{(2x-1)} + \frac{B}{(x+3)} + \frac{C}{(x+1)}$

$\Rightarrow \frac{x^2-3}{(2x-1)(x+3)(x+1)} = \frac{A(x+3)(x+1) + B(2x-1)(x+1) + C(2x-1)(x+3)}{(2x-1)(x+3)(x+1)}$

$\Rightarrow x^2-3 = A(x+3)(x+1) + B(2x-1)(x+1) + C(2x-1)(x+3)$

when $x = -3$

$\Rightarrow x^2-3 = -7B(-2)$

$\Rightarrow x^2-3 = 14B \Rightarrow 9-3 = 14B$

$\Rightarrow 6 = 14B \Rightarrow B = \frac{6}{14} = \frac{3}{7}$

when $x = -1$

$\Rightarrow 1-3 = C(-2-1)(2)$

$\Rightarrow -2 = -6C \Rightarrow C = \frac{1}{3}$

when $x = 0$

$\Rightarrow -3 = 3A - B - 3C$

$\Rightarrow -3 = 3A - \frac{3}{14} + \frac{3}{2}$

$\Rightarrow -3 = 3A - \frac{3}{2} ...$

$$\int \frac{x^2-3\,dx}{(2x-1)(x+3)(x-1)} = \int \frac{11/7}{2x-1} + \int \frac{\frac{3}{14}}{x+3} + \int \frac{-1/2}{x-1}$$

$$= \frac{11}{7}\int \frac{dx}{2x-1} + \frac{3}{14}\int \frac{dx}{x+3} - \frac{1}{2}\int \frac{dx}{x-1}$$

$$= \frac{11}{7}\cdot\frac{1}{2}\ln|2x-1| + \frac{3}{14}\ln|x+3| - \frac{1}{2}\ln|x-1|$$

Q. $$\int \frac{x^2+11x+13}{x^2-5x+6}\,dx = \int 1 + \frac{16x+7}{x^2-5x+6}\,dx$$

$$= x + I \quad \text{let}$$

let $\frac{16x+7}{x^2-5x+6} = \frac{16x+7}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{(x-3)}$

$$\Rightarrow \frac{16x+7}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$\Rightarrow 16x+7 = A(x-3) + B(x-2)$

when $x=3$

$\Rightarrow 48+7 = B \Rightarrow B = 55$

when $x=2$

$\Rightarrow 32+7 = -A$

$\Rightarrow A = -39$

$$\int \frac{16x+7}{x^2-5x+6} = \int \frac{-39}{x-2}\,dx + \int \frac{55\,dx}{x-3}$$

$$= x + -39\ln|x-2| + 55\ln|x-3| + C$$

Form : 02 Linear factor repeating

If repeating linear factor takes as many single constant as its number of repeatation.

$$\frac{5x^2+8}{(x+3)(x-6)^3} = \frac{A}{x+3} + \frac{B}{(x-6)^3} + \frac{C}{(x-6)^2} + \frac{D}{(x-6)}$$

Formula

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

Q. $$\int \frac{e^x}{e^{2x}+3e^x+1}\,dx \qquad e^x = z,\ \frac{dz}{dx} = e^x \Rightarrow dz = e^x dx$$

$$= \int \frac{dz}{z^2+3z+1}$$

$$= \int \frac{dz}{z^2+3z+\frac{9}{4}+1-\frac{9}{4}}$$

$$= \int \frac{dz}{\left(z+\frac{3}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2} = \frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\ln\left|\frac{z+\frac{3}{2}-\frac{\sqrt{5}}{2}}{z+\frac{3}{2}+\frac{\sqrt{5}}{2}}\right| + C$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{z+\frac{3}{2}-\frac{\sqrt{5}}{2}}{z+\frac{3}{2}+\frac{\sqrt{5}}{2}}\right| + C$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{e^x+\frac{3}{2}-\frac{\sqrt{5}}{2}}{e^x+\frac{3}{2}+\frac{\sqrt{5}}{2}}\right| + C$$

Q. $$\int \frac{dx}{\sin x\,(3+2\cos x)}$$

$$= \int \frac{\sin x\,dx}{\sin^2 x\,(3+2\cos x)}$$

$$= \int \frac{\sin x\,dx}{(1-\cos^2 x)(3+2\cos x)}$$

$$= -\int \frac{dz}{(1-z^2)(3+2z)} = \int \frac{dz}{(z+1)(z-1)(3+2z)}$$

Rule-03 Quadratic factor

Every quadratic factor takes a pair of constant.

eg. $$\frac{1}{(x-2)(x^2+1)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1}$$

eg. $$\frac{1}{(x^2+3)(x^2+6)} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{x^2+6}$$

(11) $$\int \frac{5x}{(x^2-2x+2)(x+1)}\,dx$$

Let $$\frac{5x}{(x^2-2x+1)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+C}$$

$$= \frac{A(x^2-2x+2)+(x+1)(Bx+C)}{(x+1)(x^2-2x+2)}$$

$\Rightarrow A(x^2-2x+2)+(x+1)(Bx+C) = 5x$

When $x=-1$, $\Rightarrow A(5) = -5$

$\Rightarrow \boxed{A=-1}$

$x=0$ $\quad 2A+C=0$

$\Rightarrow C=-2A$

$\Rightarrow \boxed{C=2}$

$x=1$, $\Rightarrow -1+2B+2C=5$

$\Rightarrow -1+2B+4=5$

$\Rightarrow B=1$

So, $$\int \frac{5x\,dx}{(x^2-2x+2)(x+1)} = \int \frac{-1\,dx}{x+1} + \int \frac{x+2\,dx}{x^2-2x+2}$$

$$= -1\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{2x-2+6}{x^2-2x+2}\,dx$$

Special problem

$$\int \frac{x^2+1}{x^4+1}\,dx$$

Integration of irrational functions :-

$\int \frac{\sqrt{x-3}}{x}\, dx$

$= \int \frac{z}{z^2+3} \cdot 2z\, dz = 2\int \frac{z^2+3-3}{z^2+3}\, dz$

$= 2\int 1 - \frac{3}{z^2+3}\, dz = 2\left[z - 3\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{z}{\sqrt{3}}\right) \right] + c$

$= 2\left[(x-3) - 3\frac{1}{\sqrt{3}} \tan^{-1} \frac{(x-3)}{\sqrt{3}} \right] + c$

$\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}}$ let $x = z^6$

$= 6\int \frac{z^5\, dz}{z^3 - z^2}$ $\Rightarrow \frac{dx}{dz} = 6z^5$

$\Rightarrow dx = 6z^5$

$= 6\int \frac{z^5\, dz}{z^2(z-1)}$

$= 6\int \frac{z^3 - 1 + 1}{z-1}\, dz = 6\int \frac{z^3-1}{z-1}\, dz + \frac{1}{z-1}\, dz$

$\int (3x+1)(x-2)^{9/2}\, dx$ let $x - 2 = z^2$

$= \int \left\{ 3(z^2+2)+1 \right\} \cdot (z^2)^{9/2} \cdot 2z\, dz$ $\Rightarrow \frac{dx}{dz} = 2z \Rightarrow dx = 2z\, dz$

$= \int (3z^2+7)\, z^{10}\, dz$

$= 2\int 3z^{12} + 7z^{10}\, dz$

$= \frac{6}{13} z^{13} + \frac{14}{11} z^{11} + c$

$= \left[\frac{6}{13} (x-2)^{13/2} + \frac{14}{11} (x-2)^{11/2} \right] + c$

$\int \sqrt{4x^2 - 4x + 5}\, dx$

$= \int \sqrt{4\left(x^2 - x + \frac{5}{4}\right)}\, dx = 2\int \sqrt{x^2 - x + \frac{1}{4} + \frac{5}{4} - \frac{1}{4}}\, dx$

$= 2\int \sqrt{\left(x - \frac{1}{2}\right)^2 + 1}\, dx$

④ $\left[\int \frac{dx}{\text{Linear}\sqrt{\text{Quadratic}}} \cdot \int \frac{dx}{L\sqrt{Q}} \right]$

For integration of the form $\int \frac{dx}{L\sqrt{Q}}$, put linear $\left(L = \frac{1}{z}\right)$

Q $\int \frac{dx}{(x-3)\sqrt{x^2+5x+27}}$

let: $x - 3 = \frac{1}{z}$

$\Rightarrow 1 = \frac{-1}{z^2}\frac{dz}{dx}$

$\Rightarrow dx = \frac{-1}{z^2}dz$

$\Rightarrow x = \frac{1}{z} + 3 = \frac{1+3z}{z}$

$= \int \frac{-1/z^2\, dz}{\frac{1}{z}\sqrt{\left(\frac{1+3z}{z}\right)^2 + 5\left(\frac{1+3z}{z}\right) + 27}}$

$= \int \frac{-1/z^2\, dz}{\frac{1}{z}\sqrt{\frac{(1+3z)^2 + 5z(1+3z) + 27z^2}{z^2}}}$

$= \int \frac{-1/z^2\, dz}{1/z^2\sqrt{9z^2+6z+1+5z+15z^2+27z^2}}$

$= -\int \frac{dz}{\sqrt{51z^2+11z+1}} = -\frac{1}{\sqrt{51}}\int \frac{dz}{\sqrt{z^2 + \frac{11}{51}z + \frac{121}{10404} + \frac{1}{51} - \ldots}}$

$= \frac{-1}{\sqrt{51}}\int \frac{dz}{\sqrt{\left(z + \frac{11}{102}\right)^2 + \alpha^2}}$

$= \frac{-1}{\sqrt{51}} \cdot \frac{\ldots}{102}$

Q $\int \frac{dx}{(2-x)\sqrt{x^2-4x+5}}$

let $2 - x = \frac{1}{z}$

$\Rightarrow -1 = \frac{-1}{z^2}\frac{dz}{dx}$

$\Rightarrow dx = \frac{1}{z^2}dz$

$= \int \frac{1/z^2\, dz}{\left(\frac{1}{z}\right)\sqrt{\left(\frac{2z-1}{z}\right)^2 - 4\left(2 - \frac{1}{z}\right) + 5}}$

Note-1 for integration of the form $\int \frac{dx}{a+b\sin x}$, $\int \frac{dx}{a+b\cos x}$, $\int \frac{dx}{a\sin x+b\cos x}$, $\int \frac{dx}{a\sin x+b\cos x + c}$

put $\boxed{\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}}$

$\boxed{\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$

Q $\int \frac{dx}{2\sin x + 3\cos x}$

$\int \frac{dx}{2\cdot\frac{2\tan x/2}{1+\tan^2 x/2} + 3\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$

$\int \frac{dx}{\frac{4\tan x/2 + 3 - \tan^2 x/2}{1+\tan^2 x/2}} = \int \frac{\sec^2 x/2 \, dx}{4\tan x/2 + 3 - 3\tan^2 x/2}$

$= \int \frac{2\,dz}{4z + 3 - 3z^2}$

$z = \tan x/2$

$\Rightarrow \frac{dz}{dx} = \frac{1}{2}\sec^2 x/2$

$\Rightarrow 2dz = \sec^2\frac{x}{2}\,dx$

$= \frac{-2}{3}\int \frac{dz}{z^2 - \frac{4}{3}z - 1}$

$= -\frac{2}{3}\int \frac{dz}{z^2 - \frac{4}{3}z + \frac{4}{9} - 1 - \frac{4}{9}} = -\frac{2}{3}\int \frac{dz}{\left(z-\frac{2}{3}\right)^2 - \left(\frac{\sqrt{13}}{3}\right)^2}$

$= \frac{-2}{3}\cdot\frac{1}{2\left(\frac{\sqrt{13}}{3}\right)} \ln\left|\frac{z - \frac{2}{3} - \frac{\sqrt{13}}{3}}{z - \frac{2}{3} + \frac{\sqrt{13}}{3}}\right| + c$

Q $\int \frac{dx}{1+\cos x+\sin x}$

$\int \frac{dx}{1 + \frac{1-\tan^2 x/2}{1+\tan^2 x/2} + \frac{2\tan x/2}{1+\tan^2 x/2}}$

$\int \frac{dx}{\frac{1+\tan^2 x/2 + 1 - \tan^2 x/2 + 2\tan x/2}{1+\tan^2 x/2}}$

$\int \frac{dx \, \sec^2 x/2}{2 + 2\tan x/2} =$ [illegible]

$z = 2 + 2\tan x/2$

$\frac{dz}{dx} = \frac{2}{2}\sec^2 x/2$

$\Rightarrow dz = \sec^2 x/2 \, dx$

$= \int \frac{dz}{z}$

$= \ln|z| + c$

$= \ln\left|2 + 2\tan x/2\right| + c$

Note-2 for integration of the form $\int \frac{dx}{a+b\sin^2 x}$, $\int \frac{dx}{a+b\cos^2 x}$, $\int \frac{dx}{a\sin^2 x+b\cos^2 x}$, $\int \frac{dx}{a\sin^2 x+b\cos^2 x+c}$,

$\int \frac{dx}{a\sin^2 x + b\cos^2 x + c\sin x\cdot\cos x + k}$

divide $\cos^2 x$ in both N^r (अंश) & D^r (हर)

Q $\int \frac{d\theta}{2+3\cos^2\theta - 4\sin^2\theta}$

$= \int \frac{\frac{1}{\cos^2\theta}\,d\theta}{\frac{2}{\cos^2\theta} + \frac{3\cos^2\theta}{\cos^2\theta} - \frac{4\sin^2\theta}{\cos^2\theta}} = \int \frac{\sec^2\theta}{2\sec^2\theta + \ldots}$

$= \int \frac{\sec^2\theta \, d\theta}{2\sec^2\theta + 3 - 4\tan^2\theta}$

$= \int \frac{\sec^2\theta \, d\theta}{2(1+\tan^2\theta)+3-4\tan^2\theta}$ $z = \tan\theta$, $\frac{dz}{d\theta} = \sec^2\theta \Rightarrow dz = \sec^2\theta\, d\theta$

$= \int \frac{dz}{2(1+z^2)+3-4z^2}$

$= \int \frac{dz}{5-2z^2} = \int \frac{dz}{(\sqrt{5})^2 - (\sqrt{2}z)^2}$

$= \frac{1}{\sqrt{2}\, 2\sqrt{5}} \ln\left| \frac{\sqrt{5}+\sqrt{2}z}{\sqrt{5}-\sqrt{2}z} \right| + C$

Note-3 for integration of the form $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}$

$N^r = A(D^r) + B(D^{r'})$

লব = A (হর) + B (হরের অবকলজ)

8. $\int \frac{3\sin x - 2\cos x}{8\sin x + 5\cos x} dx$

$3\sin x - 2\cos x = A(8\sin x + 5\cos x) + B(8\cos x - 5\sin x)$

$= 8A\sin x + 5A\cos x + 8B\cos x - 5B\sin x$

$= (8A - 5B)\sin x + (5A + 8B)\cos x$

comparing $(\times 5)\ 8A - 5B = 3$ $40A - 25B = 15$

$(\times 8)\ 5A + 8B = -2$ $40A + 64B = -16$

$-89B = 31$

$\Rightarrow B = \frac{-31}{89}$

$8A - 5B = 3$

$\Rightarrow 8A - 5\left(\frac{-31}{89}\right) = 3$

$\Rightarrow 8A + \frac{155}{89} = 3 \Rightarrow 712A + 155 = 267$

$8A = \frac{14}{89}$

So, $3\sin x - 2\cos x = \frac{14}{89}(8\sin x + 5\cos x) + \frac{-31}{89}(8\cos x - 5\sin x)$

$\Rightarrow \int \frac{3\sin x - 2\cos x}{8\sin x + 5\cos x} dx = \int \frac{14}{89} dx - \frac{31}{89} \int \frac{8\cos x - 5\sin x}{8\sin x + 5\cos x}$

$= \frac{14}{89}x - \frac{31}{89} \ln|8\sin x + 5\cos x| + C$

Note-4

form-1

For the integration of the form $\int \frac{dx}{\sin x(a+b\sin x)}$, $\int \frac{dx}{\cos x(a+b\cos x)}$ apply partial fraction

$\int \frac{dx}{\sin x(3-2\sin x)}$

Let $\frac{1}{\sin x(3-2\sin x)} = \frac{A}{\sin x} + \frac{B}{3-2\sin x} = \frac{A(3-2\sin x)+B\sin x}{\sin x(3-2\sin x)}$

$\Rightarrow A(3-2\sin x) + B\sin x = 1$

put $x = 0$, $A(3-0) + B\cdot 0 = 1$

$\Rightarrow 3A = 1 \Rightarrow A = 1/3$

$x = \pi/2$ $A(3-2) + B = 1 \Rightarrow A + B = 1 \Rightarrow B = 2/3$

So, $\int \frac{1}{\sin x(3-2\sin x)} dx = \int \frac{1/3\, dx}{\sin x} + \int \frac{2/3}{3-2\sin x} dx$

$= \frac{1}{3}\int \operatorname{cosec} x\, dx + \frac{2}{3} \int$

form-2

For the integration of the form $\int \frac{dx}{\sqrt{\sin x}(a+b\ldots}$ (free factor)

$\int \frac{dx}{\sqrt{\cos x}(a+b\sin x)}$

Multiply the free factor in both N^r and D^r and then apply partial fraction.

$\Rightarrow \frac{1}{10}\int \frac{dz}{z-1} - \frac{1}{2}\int \frac{dz}{z+1} + \frac{4}{5}\int \frac{dz}{2z+3}$

$\Rightarrow \frac{1}{10}\ln|z-1| - \frac{1}{2}\ln|z+1| + \frac{4}{10}\ln|2z+3| + c$

Note-5

For integration of the form $\frac{\sin(p \pm q)x}{\sin px \cos qx}$,

Expr: Express N^r in terms of D^r

$\int \frac{\sin(11x)}{\cos 8x \cos 3x}\,dx$

$= \int \frac{\sin(8x+3x)}{\cos 8x \cos 3x}\,dx = \int \frac{\sin 8x \cos 3x + \cos 8x \sin 3x}{\cos 8x \cdot \cos 3x}$

$= \int \frac{\sin 8x \cos 3x}{\cos 8x \cos 3x}\,dx + \int \frac{\cos 8x \sin 3x}{\cos 8x \cos 3x}$

$= \int \tan 8x\,dx + \int \tan 3x\,dx$

$= \frac{1}{8}\ln \sec 8x + \frac{1}{3}\ln \sec 3x + c$

BSE note

$\int \frac{x^2}{1+x^3}\,dx$ $\qquad z = 1+x^3$, $\frac{dz}{dx} = 3x^2$, $dz = 3x^2 dx$

$3\int \frac{dz}{1+z}$

$= 3\ln|1+x^3| + c$

$\int \frac{2\cos x}{\sin^2 x}$ $\qquad z = \sin x$, $\frac{dz}{dx} = \cos x$

$2\int \frac{dz}{z^2}$

$= 2\int z^{-2}\,dz$ $= 2\frac{z^{-2}}{-2}$ $= -\frac{\sin^{-2}}{\cdot}$

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$$= \int \frac{dz}{\sqrt{z^2 + (\sqrt{37})^2}} + 36 \int \frac{\sin 7x \, dx}{\sqrt{1 + 36 \sin^2 x}}$$

Definite Integration

fundamental Theorem of calculus (Leibnitz)

If $F(x)$ is one of anti-derivatives of

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b$$

$$= f(b) - f(a)$$

eg. $\int_2^3 \frac{1}{x}\,dx =$ $\qquad$ $\int_2^3 \frac{1}{x}\,dx$

$= \left[\ln x\right]_2^3$ $\qquad$ $\left[\ln x + 4\right]_2^3$

$= \left[\ln 3 - \ln 2\right]$ $\qquad$ $= (\ln 3 + 4) - (\ln 2 + 4)$

$= \ln 3/2$ $\qquad$ $= \ln 3 - \ln 2 = \ln 3/2$

Q. $\int_0^{\pi/2} \frac{\cos x}{1 + \sin x}\,dx$

Let $z = 1 + \sin x$

$\frac{dz}{dx} = \cos x$

$dz = \cos x$

$= \int_1^2 \frac{dz}{z}$

$x = 0, \quad z = 1$

$x = \pi/2 = 1 + \sin \pi/2$

$= 1 + 1 = 2$

$= \left[\ln z\right]_1^2 = \ln 2 - \ln 1$

$= \ln 2$

Q. $\int_0^{\ln 2} \frac{dx}{\sqrt{e^{2x} - 1}}$

Let $z = e^x$

$\frac{dz}{dx} = e^x$

$\Rightarrow dx = e^x dx$

$= \int_0^{\ln 2} \frac{e^x\,dx}{e^x \sqrt{(e^x)^2 - 1}}$

Properties

① **Dummy Variable :-**

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(t)\,dt$$

eg:
$$\int_0^{\pi} \cos 3x\,dx = \int_0^{\pi} \cos 3\theta\,d\theta$$

② $f(x) = 4x^3$

$$\int_2^3 f(x)\,dx = \int_2^3 4x^3\,dx = (x^4)_2^3 = 3^4 - 2^4 = 81 - 16 = 65$$

$$\int_2^3 f(y)\,dy = \int_2^3 4y^3\,dy = (y^4)_2^3 = 3^4 - 2^4 = 81 - 16 = 65$$

Reversal of limit :-

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a) = \lambda$$

$$\int_b^a f(x)\,dx = [f(x)]_b^a = f(a) - f(b) = -\lambda$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

③ **Breaking of the limits :-**

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

proof

$$\int_a^b f(x)\,dx = [f(x)]_a^b = f(b) - f(a)$$

$$\int_a^c f(x)\,dx = f(c) - f(a),$$

$$\int_c^b f(x)\,dx = f(b) - f(c)$$

$f(b) - f(a)$ proved

prop-05 $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sin x}{\sin x + \cos x}\,dx \quad \text{——— (I)}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sin(\pi/6 + \pi/3 - x)}{\sin(\pi/6 + \pi/3 - x) + \cos(\pi/6 + \pi/3 - x)}\,dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}\,dx \Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\cos x}{\cos x - \sin x} \quad \text{———}$$

Adding eqⁿ (I) & eqⁿ (II)

$$\Rightarrow 2I = \int_{\pi/6}^{\pi/3} dx \Rightarrow 2I = (x)_{\pi/6}^{\pi/3}$$

$$\Rightarrow I = \frac{\pi/3 - \pi/6}{2} = \pi/12$$

$$\Rightarrow \int_{\pi/6}^{\pi/3} \frac{\sin x}{\sin x + \cos x}\,dx = \pi/12$$

$$I = \int_1^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{4-x}}\,dx \quad \text{——— (1)}$$

put $x = 1+3-x$

$$= \int_1^3 \frac{\sqrt{4-x}}{\sqrt{4-x} + \sqrt{4-(4-x)}} \Rightarrow I = \int_1^3 \frac{\sqrt{4-x}}{\sqrt{4-x} + \sqrt{x}}\,dx \text{ —}$$

Adding eqⁿ (1) + (2)

$$\Rightarrow 2I = \int_1^3 1\,dx$$

$$\Rightarrow 2I = (x)_1^3 \Rightarrow I = \frac{3-1}{2} = 1$$

$$\Rightarrow \int_1^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{4-x}}\,dx = 1$$

prop - 06 (Even / odd functions)

Even

$f(-x) = f(x)$

$\cos x, \sec x, \ldots (x^2, x^4, x^6, x^8, \ldots)$

odd

$f(-x) = -f(x)$

$\sin, \tan, \cot, \operatorname{cosec} x$, [illegible], (x^3, x^5, x^7, x^9)

$o(+,-)e = o$, $o(\times \div)o = e$ — $e(+-\times\div)e = e$

$e^{odd} = o$, $o^{even} = o$ — $e^{any\ no.} = e$

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & f(x) \text{ is even} \\ 0, & f(x) \text{ is odd} \end{cases}$$

Q. $\int_{-\pi/2}^{\pi/2} \frac{\tan^3 x \cdot \operatorname{cosec}^5 x}{x \cdot \cos x}\,dx = 0$ $\left[\frac{o \cdot o}{o \cdot o} = \frac{e}{o} = o\right]$

Q. $\int_{-3}^{3} x^{101} + x^{17} + x^{27}\,dx$

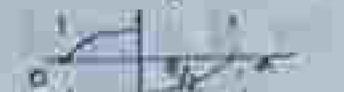
$= \int_{-3}^{3} x^{101} + x^{17}\,dx + \int_{-3}^{3} x^2\,dx$

$= 0 + 2\int_{0}^{3} x^2\,dx = 2\left[\frac{x^3}{3}\right]_{-3}^{3}$

$= 2\left(\frac{(3)^3}{3} - \frac{(-3)^3}{3}\right) = 2[9+9]$

$= 36$

prop - 07  $f(x) = f(a-x)$; $f(x)$ is sym about $x = a/2$

$f(x) = -f(a-x)$  $f(x)$ is anti sym about $x = a/2$

$$\int_0^a f(x)\,dx = \begin{cases} 2\int_0^{a/2} f(x)\,dx, & f(x) = f(a-x) \\ 0, & f(x) = -f(a-x) \end{cases}$$

prop - 8 (periodic function)

$f(x)$ is a periodic fun. with period (T)

$$\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$$

IMP Q 1- $\int_0^{\pi/2} \ln \sin x\,dx = \int_0^{\pi/2} \ln \cos x\,dx = \frac{\pi}{2}\ln\frac{1}{2}$

Q $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\,dx$

$= \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}$

$= \int_0^{\pi} \frac{(\pi - x)(-\tan x)}{(-\sec x) - \tan x}\,dx = \int_0^{\pi} \frac{(\pi - x)(\tan x)}{(\sec x + \tan x)}\,dx$

$= \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x}\,dx - \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\,dx$

$\Rightarrow 2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x}\,dx$

$\Rightarrow 2I = \pi\int_0^{\pi} \frac{\tan x (\sec x - \tan x)}{(\sec x + \tan x)(\sec x - \tan x)}$

$\Rightarrow 2I = \pi\int_0^{\pi} \frac{\tan x \sec x - \tan^2 x}{(\sec^2 x) - (\tan^2 x)}\,dx$

Integration (anti derivative) / primitive

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$

$\int \sqrt{x}\, dx =$

$= \int x^{1/2} dx = \frac{x^{3/2}}{3/2}$

$\int x^0 dx = \frac{x^1}{1} + c$

$\int 1\, dx = x + c$

$\int dx = x + c$

(2) $\int \frac{1}{x} dx = \ln x + c$

(3) $\int a^x dx = \frac{a^x}{\ln a} + c$, $a > 0$, $a \neq 1$

$\int (-3)^x = \frac{(-3)^x}{\ln}$ not defined

(4) $\int e^x dx = e^x + c$

(5) $\int \sin x\, dx = -\cos x + c$

(6) $\int \cos x\, dx = \sin x + c$

(7) $\int \tan x\, dx = \ln \sec x + c$

(8) $\int \cot x\, dx = \ln \sin$

(9) $\int \sec x\, dx = \ln |\sec x + \tan x| + c$

(10) $\int \operatorname{cosec} x\, dx = \ln |\operatorname{cosec} x - \cot x| + c$

(11) $\int \sec^2 x\, dx = \tan x + c$

(12) $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$

(13) $\int \sec x \tan x\, dx = \sec x + c$

(14) $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$

⇒ $\int u \pm v\, dx = \int u\, dx \pm \int v\, dx$

$\int \sin x + e^x dx$

$= \int \sin x\, dx + \int e^x dx$

$= -\cos x + e^x + c$

⇒ $\int k f(x)\, dx = k \int f(x)\, dx$

$\int 2x^5 dx = 2 \int \frac{x^6}{6} + c$

(i) $\int 2\, dx$

$= 2 \int dx$

$= 2x + c$

(ii) $\int 3x^2 dx$

$= 3 \int x^2 dx$

$= 3 \frac{x^3}{3} + c = x^3 + c$

(iii) $\int x^5 dx$

$= \frac{x^6}{6} + c$

(iv) $\int 2\sqrt{x} + \frac{3}{\sqrt{x}} dx$

$= 2 \int \sqrt{x}\, dx + \int \frac{3}{\sqrt{x}} dx$

$= 2 \frac{x^{3/2}}{3/2} + 3 \int \frac{1}{\sqrt{x}} dx$

$= 2 \frac{x^{3/2}}{3/2} + 3 \int x^{-1/2} dx$

$= 2 \frac{x^{3/2}}{3/2} + 3 \cdot \frac{x^{1/2}}{1/2} + c$

(v) $\int x^{4/7} + \frac{1}{x^{1/2}} dx$

$= \int x^{4/7} dx + \int \frac{1}{x^{1/2}} dx = \frac{x^{4/7+1}}{4/7+1} + \frac{x^{-1/2+1}}{-1/2+1}$

$= \int \frac{dx}{\cos^2 x} = \int \sec^2 x \, dx$

$= \tan x + c$

Q $\int \frac{dx}{1-\cos^2 x}$

$= \int \frac{1}{\sin^2 x} dx$

$= \int \operatorname{cosec}^2 x \, dx = -\cot x + c$

Q $\int \frac{2\cos x}{1-\cos^2 x} dx$

$= 2\int \frac{\cos x}{1-\cos^2 x} dx = 2\int \frac{\cos x}{\sin^2 x} dx$

~~$= 2\int \frac{\cot x}{\sin x} dx$~~

$= 2\int \cot x \cdot \operatorname{cosec} x \, dx$

$= 2(-\operatorname{cosec} x) + c$

Q $\int \frac{1-\sin^3 x}{\sin^2 x} dx = \int \frac{1}{\sin^2 x} - \frac{\sin^3 x}{\sin^2 x} dx$

$= \int \operatorname{cosec}^2 x - \sin x \, dx$

$= \int \operatorname{cosec}^2 x \, dx - \int \sin x \, dx$

$= -\cot x + \cos x + c$

Q $\int e^x + 2 \, dx = \int e^x dx + \int 2 \, dx$

$= e^x + 2x + c$

Q $\int 3^x dx$

$= \frac{3^x}{\ln 3} + c$

Q

Q $\int a^{x+2} dx = \frac{a^{x+2}}{\ln a}$

$= \int a^x \cdot a^2 dx =$ ~~$\int a^x dx \cdot \int a^2 dx$~~

$= a^2 \int a^x dx = a^2 \frac{a^x}{\ln a} + c = \frac{a^{x+2}}{\ln a} + c$

Q $\int (2x+3)^5 dx$

$= \frac{(2x+3)^6}{6 \times 2} + c$. $\left[\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c\right]$

Q $\int (3-5x)^{-4} dx$

$= \int (-5x+3)^{-4} dx = \frac{(-5x+3)^{-3}}{-3 \times -5} + c$

$= \frac{(-5x+3)^{-3}}{15} + c$

* $\int a^x dx = \frac{a^x}{\ln a} + c \longrightarrow$ * $\int a^{mx} dx = \frac{a^{mx}}{(\ln a)m}$

* $\int 3^{5x+2} dx = \frac{3^{5x+2}}{(\ln 3) 5} + c$

* $\int a^{x+2} dx$

$= \frac{a^{x+2}}{\ln a} + c$

* $\int e^x dx + c \rightarrow$ * $\int e^{mx} dx$

$= \frac{e^{mx}}{m} + c$

* $\int e^{-x} + c$

$= \frac{e^{-x}}{-1} + c$

$= -e^{-x} + c$

* $\int \sin x \, dx = -\cos x + c$ * $\int \sin mx \, dx = \frac{-\cos mx}{m} + c$

Q. $\int a^{3x}\,dx = \frac{a^{3x}}{(\ln a)3} + C$

Q. $\int \frac{e^{2x}+1}{e^x}\,dx = \int \frac{e^{2x}}{e^x} + \frac{1}{e^x}\,dx$

$= \int e^x + e^{-x}\,dx = e^x + e^{-x} + C$

(15) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$ or $-\cos^{-1}x + C$

(16) $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$ or $-\cot^{-1}x + C$

(17) $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + C$ or $-\text{cosec}^{-1}x + C$

Q. $\int \frac{5}{\sqrt{1-x^2}} + \frac{7}{1+x^2}\,dx$

$= 5\int \frac{1}{\sqrt{1-x^2}}\,dx + 7\int \frac{1}{1+x^2}\,dx = 5\sin^{-1}x + 7\tan^{-1}x$

Q. $\int \sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{1-x^2+x^2}{\sqrt{1-x^2}}\,dx$

$= \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$

Q. $\int \frac{x^2+\sqrt{x^2-1}}{x^3\sqrt{x^2-1}}\,dx = \int \frac{x^2}{x^3\sqrt{x^2-1}} + \frac{\sqrt{x^2-1}}{x^3\sqrt{x^2-1}}\,dx$

$= \int \frac{1}{x\sqrt{x^2-1}} + \frac{1}{x^3}\,dx = \int \frac{1}{x\sqrt{x^2-1}}\,dx + \int x^{-3}\,dx$

$= \sec^{-1}x + \frac{x^{-2}}{-2} + C$

Integration by substitution:

Q. $\int \sin 3x\,dx$ $\quad t = 3x \Rightarrow \frac{dt}{dx} = 3 \Rightarrow dx = \frac{dt}{3}$

$= \int \sin t \left(\frac{dt}{3}\right)$

$= \frac{1}{3}\int \sin t\,dt$

$= \frac{1}{3}(-\cos t) + C$

$= \frac{-\cos 3x}{3} + C$

Q. $\int \cos(2-7x)\,dx$ $\quad t = 2-7x \quad \frac{dt}{dx} = -7 \Rightarrow dx = \frac{dt}{-7}$

$= \int \cos t \left(\frac{dt}{-7}\right)$

~~$\int \cos x - \int \cos t \left(\frac{dt}{7}\right)$~~ $= -\frac{1}{7}\int \cos t\,dt$

$= \frac{-1}{7}\sin t + C = \frac{-1}{7}\sin(2-7x) + C$

Q. $\int \sin\frac{x}{2}\,dx$ $\quad t = \frac{x}{2} \Rightarrow \frac{dt}{dx} = \frac{1}{2} \Rightarrow dx = 2dt$

$= \int \sin t\ 2dt$

$= -2\cos t + C$

$= -2\cos\left(\frac{x}{2}\right) + C$

Q. $\int \sec^2 4x\,dx$ $\quad t = 4x \Rightarrow \frac{dt}{dx} = 4 \Rightarrow dx = \frac{dt}{4}$

$= \int \sec^2 t\left(\frac{dt}{4}\right) + C$

$= \frac{1}{4}\tan 4x + C$

$\frac{3x^2(x^2-1)}{(x^2+1)(x^2-1)} = \frac{3x^4-3x^2}{x^4-x^2+x^2-1} = \frac{3x^4-3x^2}{x^4-1}$

$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\cos x \cdot \sin x}$

Q $\int \sin x \cos x \, dx = \frac{1}{2}\int 2\sin x \cos x$

$= \frac{1}{2}\int \sin 2x \, dx = \frac{1}{2}\frac{(-\cos 2x)}{2} + C$

OR $\int t \cos x \, dx$ $\quad t = \sin x$

$\frac{dt}{dx} = \cos x \Rightarrow dt = \cos x \, dx$

$= \int t \, dt = \frac{t^2}{2} + C = \frac{\sin^2 x}{2} + C$

Q $\int \tan^3 x \sec^2 x \, dx$ $\quad t = \tan x$

$= \int t^3 \, dt$ $\quad \frac{dt}{dx} = \sec^2 x$, $dt = dx \sec^2 x$

$\Rightarrow \frac{t^4}{4} + C$

$= \frac{\tan^4 x}{4} + C$

Q $\int \frac{\sin x}{\cos^3 x} dx$ $\quad t = \cos x$

$\frac{dt}{dx} = -\sin x$

$= \int \frac{-dt}{t^3}$ $\quad \Rightarrow -dt = \sin x \, dx$

$= -\int t^{-3} dt = -\left(\frac{t^{-2}}{-2} + C\right)$

$= \frac{-(\cos x)^{-2}}{-2} + C$

Q $\int \frac{\operatorname{cosec}^2(\ln x)}{x} dx$ $\quad t = \ln x$

$= \int \frac{\operatorname{cosec}^2 t}{x} dt\, x$ $\quad \Rightarrow \frac{dt}{dx} = \frac{1}{x}$

$\Rightarrow dx = dt\, x$

$= -\cot t + C$

$= -\cot(\ln x) + C$

$\frac{-1}{2}^{+1} = \frac{-1+2}{2}$

Q $\int \frac{\operatorname{cosec}^2 x}{1+\cot x} dx$ $\quad t = 1+\cot x$

$\frac{dt}{dx} = -\operatorname{cosec}^2 x$

$-dt = \operatorname{cosec}^2 x \, dx$

$= \int \frac{\operatorname{cosec}^2 x}{t}$ ~~$\operatorname{cosec}^2 x \, dx$~~

$= \int \frac{-dt}{t} = -\int \frac{1}{t} dt$

$= -\ln t + C$

$= -\ln|1+\cot x| + C$

Q $\int \sqrt{1-\sin x}\, \cos x \, dx$ $\quad t = 1-\sin x$

$\frac{dt}{dx} = -\cos x$

$-dt = \cos x \, dx$

$= \int t^{1/2} \cdot -dt$

$= -\left(\frac{t^{3/2}}{3/2}\right) = -\frac{2}{3}(1-\sin x)^{3/2} + C$

* Formula

→ $\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$

→ $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

→ $\sin A \sin A = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

Q $\int \sin 4x \cos 3x \, dx$

$= \frac{1}{2}\int \sin(4x-3x) + \sin(4x+3x) \, dx$

$= \frac{1}{2}\int \sin x + \sin 7x \, dx$

$= \frac{1}{2}\left[-\cos x - \frac{\cos 7x}{7}\right] + C$

Q $\int \cos 5x \cos 2x \, dx$

$= \frac{1}{2}\int \cos 3x + \cos 7x$

$= \frac{1}{2}\left[\frac{\sin 3x}{3} + \frac{\sin 7x}{7}\right] + C$

Q $\int \sin x \cos 4x \, dx = \frac{1}{2}\int \sin(-3x) + \sin(5x)$

$= \frac{1}{2}\left[\frac{\cos 3x}{3} - \frac{\cos 5x}{5}\right] + C$

Q $\int \sin 6x \sin 3x \, dx = \frac{1}{2}\int \cos 3x - \cos 9x$

$= \frac{1}{2}\left[\frac{\sin 3x}{3} - \frac{\sin 9x}{9}\right] + C$

Q $\int \cos 3x \sin 4x \, dx = \frac{1}{2}\int \sin x + \sin 7x$

$= \frac{1}{2}\left[-\cos x + \frac{-\cos 7x}{7}\right] + C$

$= \frac{1}{2}\left[-\cos x - \frac{\cos 7x}{7}\right] + C$

Q $\int \cos 4x \cos 5x \sin 2x \, dx$

$= \frac{1}{2}\int \left[\cos(4x - 5x) + \cos(4x + 5x)\right] \sin 2x \, dx$

$= \frac{1}{2}\int \left[\cos(-x) + \cos 9x\right] \sin 2x \, dx$

$= \frac{1}{2}\int \cos 2x \cos x + \sin 2x \cos 9x \, dx$

$= \frac{1}{2}\int \frac{1}{2}\left[\sin x + \sin 3x\right] + \frac{1}{2}\left[\sin(-7x) + \sin 11x\right] dx$

$= \frac{1}{4}\int \sin x + \sin 3x - \sin 7x + \sin 11x \, dx$

$= \frac{1}{4}\left[-\cos x - \frac{\cos 3x}{3} + \frac{\cos 7x}{7} - \frac{\cos 11x}{11}\right] + C$

(3) $\frac{C}{\cdot}$

Form = 02 $\cos^2 2x = \frac{1 + \cos 4x}{2}$

$\rightarrow \int \sin^2 x \, dx, \int \sin^4 x \, dx$

$\sin^2 x = \frac{1 - \cos 2x}{2}$	$\sin^4 x =$
$\cos^2 x = \frac{1 + \cos 2x}{2}$	$\cos^4 x =$

Q $\int \cos^2 x = \int \frac{1 + \cos 2x}{2} dx$

$= \frac{1}{2}\int 1 + \cos 2x \, dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + C$

Q $\int \sin^2 x = \int \frac{1 - \cos x}{2} = \frac{1}{2}\int 1 - \cos 2x$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$

Q $\int \cos^4 x \, dx = \int (\cos^2 x)^2 dx$

$= \int \left(\frac{1 + \cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 + \cos 2x)^2 dx$

$= \frac{1}{4}\int 1 + \cos^2 2x + 2\cos 2x \, dx$

$= \frac{1}{4}\int 1 + \frac{1 + \cos 4x}{2} + 2\cos 2x \, dx$

$= \frac{1}{4}\int \frac{2 + 1 + \cos 4x + 4\cos 2x}{2} dx$

$= \frac{1}{8}\int 3 + \cos 4x + 4\cos 2x \, dx$

$= \frac{1}{8}\left[3x + \frac{\sin 4x}{4} + \frac{4\sin 2x}{2}\right] + C$

Q $\int \sin^4 x \, dx = \int (\sin^2 x)^2 dx$

$= \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - \cos 2x)^2 dx$

$= \frac{1}{4}\int 1 + \cos^2 2x - 2\cos 2x \, dx$

$= \frac{1}{4}\int 1 + \frac{1 + \cos 4x}{2} - 2\cos 2x \, dx$

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$= \frac{1}{4}\int \frac{2+1+\cos 4x}{2} - 4\cos 2x \, dx$

$= \frac{1}{8}\int 3 + \cos 4x - 4\cos 2x \, dx$

$= \frac{1}{8}\left[3x + \frac{\sin 4x}{4} - \frac{4\sin 2x}{2}\right] + c$

Form - 03

$\int \sin^3 x \, dx$, $\int \cos^3 x \, dx$

$\int \sin^5 x \, dx$ $\quad \int \cos^5 x \, dx$

$\int \sin^7 x \, dx$ $\quad \int \cos^7 x \, dx$

Q. $\int \sin^3 x \, dx = \int \sin^2 x \cdot \sin x \, dx$

$= \int (1-\cos^2 x) \sin x \, dx$ $\qquad t = \cos x$, $\frac{dt}{dx} = -\sin x$ $\Rightarrow -dt = \sin x \, dx$

$= \int (1-t^2)(-dt)$

$= \int t^2 - 1 \, dt$

$= \frac{t^3}{3} - t + c = \frac{\cos^3 x}{3} - \cos x + c$

Q. $\int \cos^3 x \, dx = \int \cos^2 x \cdot \cos x \, dx$ $\qquad t = \sin x$, $\frac{dt}{dx} = \cos x$, $dt = \cos x \, dx$

$= \int (1-\sin^2 x) \cdot \cos x \, dx$

$= \int (1-t^2) \, dt$

$= \int t^2 - 1 \, dt$

$= \left[\frac{t^3}{3} - t\right] + c = \left(\frac{\sin^3 x}{3} - \sin x + C\right)$

Q. $\int \sin^5 x \, dx = \int \sin^4 x \sin x \, dx$

$= \int (\sin^2 x)^2 \cdot \sin x \, dx$

$= \int (1-\cos^2 x)^2 \sin x \, dx$ $\qquad t = \cos x$, $\frac{dt}{dx} = -\sin x$ $\Rightarrow -dt = \sin x \, dx$

$= \int (1-t^2)^2 (-dt)$

$= -\int 1 + t^4 - 2t^2 \, dt$

$= -\left[t + \frac{t^5}{5} - \frac{2t^3}{3}\right] + c$

$= -\left[\cos x + \frac{\cos^5 x}{5} - \frac{2\cos^3 x}{3}\right] + c$

Q. $\int \cos^5 x \, dx = \int \cos^4 x \cdot \cos x \, dx$ $\qquad t = \sin x$, $\frac{dt}{dx} = \cos x$, $dt = \cos x \, dx$

$\int (\cos^2 x)^2 \cdot \cos x \, dx$

$= \int (1-\sin^2 x)^2 \cdot \cos x \, dx$

$= \int (1-t^2)^2 \cdot (dt)$

$= \int 1 + t^4 - 2t^2 \, dt$

$= \left[t + \frac{t^5}{5} - \frac{2t^3}{3}\right]$

$= \left[\sin x + \frac{\sin^5 x}{5} - \frac{2\sin^3 x}{3}\right]$

Form - 04

$\int \sin^m x \cos^n x \, dx$

or $\int \frac{\sin^m x}{\cos^n x} \, dx$

Case-1 when both are even

$\int \sin^2 x \cos^2 x \, dx$

$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx = \frac{1}{4}\int 1 - \cos^2 2x \, dx$

$= \frac{1}{4}\int \sin^2 2x \, dx$

$= \frac{1}{4}\int \frac{1-\cos 4x}{2} \, dx = \frac{1}{8}\int 1 - \cos 4x \, dx$

$= \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right] + c$

Case-I when both are odd :

$\int \sin^5 x \cos^3 x \, dx$

$= \int \sin^5 x \cos^2 x \cos x \, dx = \int \sin^5 x (1 - \sin^2 x) \cos x \, dx$

$= \int \sin^5 x (1 - \sin^2 x) \cos x \, dx$

$t = \sin x$

$\frac{dt}{dx} = \cos x$

$\Rightarrow dt = \cos x \, dx$

$= \int t^5 (1 - t^2) \, dt$

$= \int t^5 - t^7 \, dt$

$= \frac{t^6}{6} - \frac{t^8}{8} + C = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C$

Case-II when one is even & other is odd

$\int \sin^4 x \cos^3 x \, dx$

$= \int \sin^4 x \cos^2 x \, dx$

$= \int \sin^4 x (1 - \sin^2 x) \cos x \, dx$

$t = \sin x$

$\frac{dt}{dx} = \cos x$

$\Rightarrow dt = \cos x \, dx$

$= \int t^4 (1 - t^2) \, dt$

$= \int t^4 - t^6 \, dt$

$= \frac{t^5}{5} - \frac{t^7}{7} + C = \frac{\sin^5}{5} - \frac{\sin^7}{7} + C$

Q. $\int \sin^3 x \, dx$

$\sin 3x = 3\sin x - 4\sin^3 x$

$\Rightarrow 4\sin^3 x = 3\sin x - \sin 3x$

$\Rightarrow \sin^3 x = \frac{3\sin x - \sin 3x}{4}$

$= \int \frac{3\sin x - \sin 3x}{4}$

$= \frac{1}{4} \int 3\sin x - \sin 3x \, dx$

$= \frac{1}{4} \left[3(-\cos x) + \frac{\cos 3x}{3} \right] + C$

Q. $\int \cos^3 x \, dx$

$\cos 3x = 4\cos^3 x - 3\cos x$

$\Rightarrow \frac{\cos 3x + 3\cos x}{4} = \cos^3 x$

$= \int \frac{\cos 3x + 3\cos x}{4} \, dx$

$= \frac{1}{4} \int \cos 3x + 3\cos x \, dx = \frac{1}{4} \left[\frac{\sin 3x}{3} + 3\sin x \right] + C$

(2) (3) $\int \sin^4 x \cos^4 x \, dx$

$= \int \sin^4 x$:

Q. $\int \sin^6 x \, dx$

$\int (\sin^3 x)^2 \, dx$

$\sin 3x = 3\sin x - 4\sin^3 x$

$\Rightarrow \sin^3 x = \frac{3\sin x - \sin 3x}{4}$

$= \int \left(\frac{3\sin x - \sin 3x}{4} \right)^2 dx$

$= \frac{1}{16} \int 9\sin^2 x + \sin^2 3x - 6\sin 3x \sin x \, dx$

$= \frac{1}{16} \left[9 \int \frac{1 - \cos 2x}{2} \, dx + \int \frac{1 - \cos 6x}{2} \, dx - \frac{6}{2} \int \cos 2x - \cos 4x \, dx \right]$

$= \frac{1}{16} \left[\frac{9}{2} \left(x - \frac{\sin 2x}{2} \right) + \frac{1}{2} \left(x - \frac{\sin 6x}{6} \right) - 3 \left(\frac{\sin 2x}{2} - \frac{\sin 4x}{4} \right) \right] + C$

Q. $\int \tan^5 \theta \sec^4 \theta \, d\theta$

$= \int \tan^5 \theta \sec^2 \theta \sec^2 \theta \, d\theta$

$t = \tan \theta$

$= \int \tan^5 \theta (1 + \tan^2 \theta)(\sec^2 \theta)$

$\Rightarrow \frac{dt}{d\theta} = \sec^2 \theta$

$= \int t^5 (1 + t^2) \, dt = \int t^5 + t^7 \, dt$

$\Rightarrow dt = \sec^2 \theta \, d\theta$

$= \frac{t^6}{6} + \frac{t^8}{8} + C = \frac{\tan^6 \theta}{6} + \frac{\tan^8 \theta}{8} + C$

Q. $\int \cot^4 \theta \operatorname{cosec}^4 \theta \, d\theta$

$t = \cot \theta$

$\Rightarrow \frac{dt}{d\theta} = -\operatorname{cosec}^2 \theta$

$\Rightarrow -dt = \operatorname{cosec}^2 \theta \, d\theta$

$= \int \cot^4 \theta \operatorname{cosec}^2 \theta \operatorname{cosec}^2 \theta \, d\theta$

$= \int \cot^4 \theta (1 + \cot^2 \theta) \operatorname{cosec}^2 \theta \, d\theta$

$= \int \cot^4 \theta (1 + t^2)(-dt)$

$= -\int (t^4 + t^6)\, dt$

$= -\frac{t^5}{5} + \frac{t^7}{7} + C = -\frac{\cot^5\theta}{5} + \frac{\cot^7\theta}{7} + C$

Q. $\int \sec^{11}\theta \tan\theta \, d\theta$

$= \int \sec^{10}\theta \cdot \sec\theta \cdot \tan\theta \, d\theta$

$\int \sec^{2n+1}\theta \tan^{2n+1}\theta$

$t = \sec\theta$

$t = \sec\theta$

$\Rightarrow \frac{dt}{d\theta} = \sec\theta \cdot \tan\theta$

$\Rightarrow dt = \sec\theta \cdot \tan\theta \, d\theta$

$\Rightarrow \int t^{10} \cdot dt$

$= \frac{t^{11}}{11} + C$

$= \frac{\sec^{11}\theta}{11} + C$

Q. $\int \cot\theta \operatorname{cosec}^7\theta \, d\theta$

$t = \operatorname{cosec}\theta$

$\Rightarrow \frac{dt}{d\theta} = -\operatorname{cosec}\theta \cdot \cot\theta$

$\Rightarrow -dt = \operatorname{cosec}\theta \cdot \cot\theta \, d\theta$

$= \int \operatorname{cosec}^6\theta \cdot \operatorname{cosec}\theta \cdot \cot\theta$

$= \int t^6 (-dt)$

$= -\left[\frac{t^7}{7} + C\right] = -\left[\frac{\operatorname{cosec}^7\theta}{7} + C\right]$

FORM NO - 05

* $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

Q. $\int \frac{dx}{\sqrt{4 - x^2}} = \int \frac{dx}{\sqrt{2^2 - x^2}} = \sin^{-1}\frac{x}{2} + C$

Q. $\int \frac{dx}{\sqrt{5 - x^2}} = \int \frac{dx}{\sqrt{(\sqrt{5})^2 - x^2}} = \sin^{-1}\frac{x}{\sqrt{5}} + C$

Q. $\int \frac{dx}{\sqrt{9 - 4x^2}} = \int \frac{dx}{\sqrt{3^2 - (2x)^2}} = \frac{1}{2}\sin^{-1}\frac{2x}{3} + C$

let $t = 2x$

$\frac{dt}{dx} = 2 \Rightarrow dx = \frac{dt}{2}$

$= \frac{1}{2}\int \frac{dt}{\sqrt{3^2 - t^2}}$

$= \frac{1}{2}\int \frac{dt}{\sqrt{3^2 - t^2}} = \frac{1}{2}\sin^{-1}\frac{t}{3} + C$

$= \frac{1}{2}\left[\sin^{-1}\frac{2x}{3} + C\right]$

Q. $\int \frac{dx}{\sqrt{11 - 4x^2}} = \int \frac{dx}{\sqrt{(\sqrt{11})^2 - (2x)^2}}$

$= \frac{1}{2}\left[\sin^{-1}\frac{2x}{\sqrt{11}} + C\right]$

Q. $\int \frac{e^{3x}\, dx}{\sqrt{4 - e^{6x}}}$

$= \int \frac{e^{3x}\, dx}{\sqrt{(2)^2 - (e^{3x})^2}}$

$t = e^{3x}$

$\frac{dt}{dx} = e^{3x} \cdot 3$

$\frac{dt}{3} = e^{3x} dx$

$= \int \frac{\frac{dt}{3}}{\sqrt{(2)^2 - (e^{3x})^2}}$

$= \frac{1}{3}\int \frac{dt}{\sqrt{2^2 - (e^{3x})^2}} = \frac{1}{3}\left[\sin^{-1}\frac{e^{3x}}{2} + C\right]$

Q. $\int \frac{dx}{x\sqrt{25 - (\ln x)^2}}$

$t = \ln x$

$\frac{dt}{dx} = \frac{1}{x}$

$\Rightarrow dt\, x = dx$

$= \int \frac{dx}{x\sqrt{(5)^2 - (\ln x)^2}}$

$= \int \frac{dt\, x}{x\sqrt{(5)^2 - (t)^2}} = \int \frac{dt}{\sqrt{(5)^2 - (t)^2}}$

$= \sin^{-1}\left(\frac{\ln x}{5}\right) + C$

Q. $\int \frac{\cos\theta}{\sqrt{4 - \sin^2\theta}} d\theta = \int \frac{\cos\theta}{\sqrt{(2)^2 - (\sin\theta)^2}} d\theta$

$t = \sin\theta$

$\frac{dt}{d\theta} = \cos\theta$

$\Rightarrow \frac{dt}{\cos\theta} = d\theta$

$= \int \frac{\cos\theta}{\sqrt{2^2 - (t)^2}} \cdot \frac{dt}{\cos\theta}$

$= \int \frac{dt}{\sqrt{(2)^2 - t^2}} = \sin^{-1}\frac{t}{2} + C$

$= \sin^{-1}\frac{\sinh\theta}{2} + C$

* $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

Ex. $\int \frac{dx}{4 + x^2} = \int \frac{dx}{(2)^2 + (x)^2}$

$= \frac{1}{2}\tan^{-1}\frac{x}{2} + C$

Ex. $\int \frac{dx}{2x^2 + 7} = \int \frac{dx}{(\sqrt{7})^2 + (\sqrt{2}x)^2}$

$= \frac{1}{\sqrt{2}\sqrt{7}}\tan^{-1}\frac{\sqrt{2}x}{\sqrt{7}} + C$

Ex. $\int \frac{e^{4x}}{e^{8x} + 4} = \int \frac{e^{4x}}{(e^{4x})^2 + 4}\,dx$

$= \int \frac{e^{4x}}{(2)^2 + (t)^2}\frac{dt}{4e^{4x}}$ $\quad t = e^{4x}$, $\frac{dt}{dx} = 4e^{4x}$

$= \frac{1}{4}\int \frac{dt}{(2)^2 + t^2}$ $\quad \Rightarrow dx = \frac{dt}{4e^{4x}}$

$= \frac{1}{4}\frac{1}{2}\tan^{-1}\frac{t}{2} + C$

$= \frac{1}{8}\tan^{-1}\frac{e^{4x}}{2} + C$

Ex. $\int \frac{dx}{x[(\ln x)^2 + 25]} = \int \frac{dx}{x(5)^2 + (\ln x)^2}$ $\quad t = \ln x$

$= \int \frac{dt\,x}{x(5)^2 + (t)^2}$ $\quad \frac{dt}{dx} = \frac{1}{x}$

$= \int \frac{dt}{(5)^2 + t^2}$ $\quad \Rightarrow dx = dt\,x$

$= \frac{1}{5}\tan^{-1}\frac{t}{5} + C = \frac{1}{5}\tan^{-1}\frac{\ln x}{5} + C$

* $\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1}x + C$

* $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

Ex. $\int \frac{dx}{x\sqrt{4x^2 - 9}} = \int \frac{dx}{x\sqrt{(2x)^2 - (3)^2}}$ $\quad t = 2x$, $\Rightarrow \frac{dt}{dx} = 2$, $\Rightarrow \frac{1}{2}dt = dx$

$= \int \frac{dt}{2x\sqrt{(2x)^2 - (3)^2}}$

$= \int \frac{dt}{t\sqrt{t^2 - 3^2}}$

$= \frac{1}{3}\sec^{-1}\frac{t}{3} + C = \frac{1}{3}\sec^{-1}\frac{2x}{3} + C$

Ex. $\int \frac{dx}{\sqrt{e^{4x} - 5}} = \int \frac{dx}{\sqrt{(e^{2x})^2 - (\sqrt{5})^2}}$ $\quad t = e^{2x}$, $\frac{dt}{dx} = 2e^{2x}$, $\Rightarrow dx = \frac{dt}{2e^{2x}}$

$= \int \frac{dt}{2e^{2x}\sqrt{(t)^2 - (\sqrt{5})^2}}$

$= \int \frac{dt}{2t\sqrt{t^2 - (\sqrt{5})^2}} = 2\int \frac{dt}{t\sqrt{t^2 - (\sqrt{5})^2}}$

$= 2\frac{1}{\sqrt{5}}\sec^{-1}\frac{e^{2x}}{\sqrt{5}} + C$

* $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left|x + \sqrt{a^2 + x^2}\right| + C$

Ex. $\int \frac{dx}{\sqrt{x^2 + 4}} = \ln\left|x + \sqrt{x^2 + 4^2}\right| + C$

Ex. $\int \frac{dx}{\sqrt{3x^2 + 4}} = \int \frac{dx}{(\sqrt{3}x)^2 + (2)^2}$

$= \frac{1}{\sqrt{3}}\ln\left|\sqrt{3}x + \sqrt{(\sqrt{3}x)^2 + (2)^2}\right| + C$

Q. $\int \frac{e^x \cos e^x \, dx}{\sin^2 e^x + 9}$ $\int \frac{e^x \cos e^x \, dx}{\sin^2 e^x + 9}$

$t = \sin e^x$

$\Rightarrow \frac{dt}{dx} = \cos e^x \cdot e^x$

$\Rightarrow dt = \cos e^x \cdot e^x$

$= \int \frac{dt}{(t)^2 + (3)^2}$

$= \ln \left| x + \sqrt{t^2 + 3^2} \right| + C$

$= \ln \left| \sin e^x + \sqrt{\sin^2 e^x + 3^2} \right| + C$

* $\boxed{\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + C}$

① $\int \frac{dx}{\sqrt{4x^2 - 6}}$ $\int \frac{dx}{\sqrt{(2x)^2 - (\sqrt{6})^2}}$

$= \frac{1}{2}\left[\sin^{-1} \frac{2x}{\sqrt{6}} + C\right]$

② $\int \frac{dx}{\sqrt{6 - 4x^2}} = \int \frac{dx}{\sqrt{(\sqrt{6})^2 - (2x)^2}}$

$t = 2x$

$\Rightarrow \frac{dt}{dx} = 2$

$\Rightarrow dx = \frac{dt}{2}$

$= \int \frac{\frac{dt}{2}}{\sqrt{(\sqrt{6})^2 - (t)^2}}$

$= \frac{1}{2} \int \frac{dt}{\sqrt{(\sqrt{6})^2 - (t)^2}} = \frac{1}{2} \ln \left| 2x + \sqrt{(2x)^2 - (\sqrt{6})^2} \right| + C$

Integration by parts :-

I - Inv. trigonometric function $\sin^{-1}x, \cos^{-1}x$

L - logarithmic function

A - Algebraic function

T - Trigonometric function

E - Exponential function e^x, a^x, e^x

$\int uv = \int u \cdot e^x$

* $\boxed{\int uv = u \int v \, dx - \int \left[\frac{d}{dx} u \int v \, dx \right] dx}$

Q. $\int x e^x \, dx$

$= x \int e^x dx - \int \left[\frac{d}{dx} x \int e^x \, dx \right] dx$

$= x e^x - \int 1 \cdot e^x \, dx$

$= x e^x - e^x + C$

Q. $\int x \cos x$

$= x \int \cos x \, dx - \int \left[\frac{d}{dx} x \int \cos x \, dx \right] dx$

$= x \sin x - \int (1 \cdot \sin x \, dx)$

$= + x \sin x - (-\cos x) + C$

$= x \sin x + \cos x + C$

Q. $\int (1+x) e^x \, dx$

$(1+x) \int e^x \, dx - \int \left[\frac{d}{dx}(1+x) \int e^x \, dx \right] dx$

$= (1+x) e^x - \int 1 \cdot e^x \, dx$

$= (1+x) e^x - e^x + C$

Q. $\int x \cos^2 x \, dx$

$x \int \cos^2 x \, dx - \int \left[\frac{d}{dx} x \int \cos^2 x \, dx \right] dx$

$= x \left(\frac{1 + \cos 2x}{2} \right) - \int 1 \left(\frac{1 + \cos 2x}{2} \right) dx$

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$= \int \frac{(x^3)^2 + 1 - 1}{x^2+1} dx = \int \frac{(x^3)^2+1}{x^2+1} - \frac{1}{x^2+1} dx$

$= \int \frac{(x^2+1)(x^4 - x^2 + 1)}{(x^2+1)} - \frac{1}{(x^2+1)} dx$

$= \int (x^4 - x^2 + 1) - \frac{dx}{x^2+1}$

Q. $\int \frac{x^4 + x^3 + x^2 + x + 2}{x^2+1} dx$

$x^2+1 \,)\, x^4 + x^3 + x^2 + x + 2 \,(\, x^2 + x$
$x^4 + x^2$
$x^3 + x + 2$
$x^3 + x$
$+2$

$= \int x^2 + x + \frac{2}{x^2+1} dx$

$= \int x^2 + \int x + 2\int \frac{dx}{x^2+1}$

$= \frac{x^3}{3} + \frac{x^2}{2} + 2\tan^{-1} x + c$

or $\int \frac{x^4 + x^3 + x^2 + x + 2}{x^2+1} dx$

$= \int \frac{x^4 + x^2 + x^3 + x}{x^2+1} dx = \int \frac{x^2(x^2+1) + x(x^2+1) + 2}{x^2+1} dx$

Form :- 10

$$\int [f(x)]^n \cdot f'(x) \cdot dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

Q. $\int \sin^5 x \cos x \, dx$

$= \frac{\sin^6 x}{6} + c$

Q. $\int x\sqrt{x^2+3}\, dx = \frac{1}{2}\int (x^2+3)^{1/2}\, 2x \, dx$

$= \frac{1}{2} \frac{(x^2+3)^{3/2}}{3/2} + c$

$= \int (x^4 - 3x^2 + 1)^4 \cdot (2x^3 - 3x) dx$

$= \frac{1}{2} \frac{(x^4 - 3x^2 + 1)^5}{5} + c$

Form :- 11

$$\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$$

Q. $\int \frac{\cos x}{\sqrt{\sin x}} dx = 2\sqrt{\sin x} + c$

Q. $\int \frac{x}{\sqrt{x^2 - a^2}} dx$

$= \frac{1}{2} \int \frac{2x}{\sqrt{x^2 - a^2}} = \frac{1}{2} \times 2\sqrt{x^2 - a^2} + c$

Form :- 12

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$$

Q. $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$

$= \ln \sin x + c$

Q. $\int \frac{\operatorname{cosec}^2 x}{1 + \cot x} dx = -\int \frac{-\operatorname{cosec}^2 x}{1 + \cot x} dx$

$= -\ln |1 + \cot x| + c$

Q. $\int \frac{3x+4}{x^2+4} dx$

$= \int \frac{3x}{x^2+4} dx + \int \frac{4}{x^2+4} dx$

$= 3\int \frac{x}{x^2+4} dx + 4\int \frac{1}{x^2+2^2} dx$

$= \frac{3}{2} \int \frac{2x}{x^2+4} dx + 4 \times \frac{1}{2} \tan^{-1} \frac{x}{2} + c$

Form

* $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

* $\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C$

* $\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

(1)(i) $\int\sqrt{9-x^2}\,dx = \int\sqrt{(3)^2-(x)^2}$

$= \frac{x}{3}\sqrt{(3)^2-(x)^2} + \frac{(3)^2}{2}\sin^{-1}\frac{x}{3} + C$

(ii) $\int\sqrt{16-x^2}\,dx = \int\sqrt{(4)^2-(x)^2}$

$= \frac{x}{4}\sqrt{(4)^2-(x)^2} + \frac{(4)^2}{2}\sin^{-1}\frac{x}{4} + C$

(iii) $\int\sqrt{5-(4x)^2}\,dx = \int\sqrt{(\sqrt{5})^2-(2x)^2}\,dx$ $\quad t = 2x$, $\frac{dt}{dx} = 2$

$= \frac{1}{2}\int\sqrt{(\sqrt{5})^2-(t)^2}\,dt$ $\quad \Rightarrow dx = \frac{dt}{2}$

$= \frac{1}{2}\cdot\frac{t}{2}\sqrt{(\sqrt{5})^2-(t)^2} + \frac{(\sqrt{5})^2}{2}\sin^{-1}\frac{t}{\sqrt{5}} + C$

$= \frac{2x}{2}\sqrt{(\sqrt{5})^2-(2x)^2} + \frac{(\sqrt{5})^2}{4}\sin^{-1}\frac{2x}{\sqrt{5}} + C$

(iv) $\int e^z\sqrt{4-e^{2z}}\,dz$ $\quad t = e^z$

$= \int e^z\sqrt{(2)^2-(e^z)^2}\,dz$ $\quad \Rightarrow \frac{dt}{dz} = e^z$, $\Rightarrow dt = e^z\,dz$

$= \int\sqrt{(2)^2-(t)^2}\,dt$

$= \frac{e^z}{2}\sqrt{(2)^2-(e^z)^2} + 2\sin^{-1}\frac{e^z}{2} + C$

(v) $\int\sqrt{x^2+4}\,dx$

$= \int\sqrt{x^2+(2)^2} = \frac{x}{2}\sqrt{x^2+2^2} + \frac{2^2}{2}\ln\left|x+\sqrt{x^2+2^2}\right| + C$

Form

$$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$$

Q $\int e^x [\sin x + \cos x]\, dx = e^x \sin x + C$

Q $\int e^x [\tan x + \ln \sec x] = e^x \ln|\sec x| + C$

Q $\int e^x [1 + x \ln x] + C = e^x \ln|x \ln x| + C$

Q $\int \frac{e^x}{x} [1 + x \ln x] + C = \int e^x \left[\frac{1}{x} + \ln x\right] dx = e^x \ln x + C$

Differential equation:-

Order & Degree :-

Differential equation

Formation   Solution

Formation of differential equation
(By elliminating orbitrary constant)

Type - 01

Q
$y = A\sec x$

$y_1 = A\sec x \cdot \tan x$

$\Rightarrow y_1 = y\tan x$

$\Rightarrow y_1 - y\tan x = 0$

Q
$y = A\sin x \Rightarrow A = y/\sin x$

$y_1 = A\cos x$

$y_1 = y/\sin x \cdot \cos x$

$\Rightarrow y_1 = y\dfrac{\cos x}{\sin x}$ $\Rightarrow y_1 = y\cot x$

$\Rightarrow y_1 - y\cot x = 0$

Q
Type - 02

$y = Ae^t + Be^{2t}$

$\Rightarrow y_1 = Ae^t + B2e^{2t}$

$\Rightarrow y_2 = Ae^t + B2e^{2t} \times 2$

$\Rightarrow y_2 = Ae^t + B4e^{2t}$ $\Rightarrow Ae^t + B4e^{2t} - y_2 = 0$

$$\begin{vmatrix} e^t & e^{2t} & -y \\ e^t & 2e^{2t} & -y_1 \\ e^t & 4e^{2t} & -y_2 \end{vmatrix} = 0$$

$\Rightarrow -e^{x}e^{2x}\begin{vmatrix} 1 & 1 & y \\ 1 & 2 & y_1 \\ 1 & 4 & y_2 \end{vmatrix} = 0$

$\Rightarrow 1(2y_2 - 4y_1) - 1(y_2 - y_1) + y(4-2) = 0$

$\Rightarrow 2y_2 - 4y_1 - y_2 + y_1 + 4y - 2y = 0$

$\Rightarrow y_2 - 3y_1 + 2y = 0$

$y = a\sin^{-1}x + b\cos^{-1}x$

$\Rightarrow y_1 = \frac{a}{\sqrt{1-x^2}} - \frac{b}{\sqrt{1-x^2}}$

$\Rightarrow y_1 = \frac{a-b}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\, y_1 = a-b$

$\Rightarrow \sqrt{1-x^2}\, y_2 + \frac{y_1}{2\sqrt{1-x^2}} \cdot (-2x) = 0$

$\Rightarrow \frac{(1-x^2)y_2 - xy_1}{\sqrt{1-x^2}} = 0$

$\Rightarrow (1-x^2)y_2 - xy_1 = 0$

Solution of differential equation :-

- General solⁿ
- Particular solⁿ

General solution :-

- Variable separation method
- Homogeneous differential equation
- Linear diff. eqⁿ.

Type - 01 Solve $\frac{dy}{dx} = f(x)$

$\Rightarrow dy = f(x)\,dx$

$\Rightarrow \int dy = \int f(x)\,dx$

$\frac{dy}{dx} = \frac{e^{2x}+1}{e^{x}}$

$\Rightarrow dy = \frac{e^{2x}+1}{e^{x}}\,dx \Rightarrow \int dy = \int \frac{e^{2x}+1}{e^{x}}\,dx$

$\Rightarrow \int dy = \int e^{x} + e^{-x} \Rightarrow y = e^{x} - e^{-x} + c$

Solve $\frac{dy}{dx} = x\cos x$

$\Rightarrow \int dy = \int x\cos x\,dx$

$\Rightarrow y = x\int \cos x\,dx - \int\left[\frac{d}{dx}x \int \cos x\,dx\right]dx$

$\Rightarrow y = x\sin x - \int 1 \cdot \sin x\,dx$

$\Rightarrow y = x\sin x + \cos x + c$

Type - 02

Solve $\frac{dy}{dx} = f(y)$

$\Rightarrow \int dx = \int \frac{dy}{f(y)}$

$\Rightarrow$ Solve $\frac{dy}{dx} = y+2$

$\Rightarrow \int \frac{dy}{y+2} = \int dx \Rightarrow$

$\ln|y+2| = x + c$

Type - 03

Solve $\frac{dy}{dx} = f(x)\,f(y)$ or $\frac{dy}{dx} = \frac{f(x)}{f(y)}$

$\Rightarrow \int \frac{dy}{f(y)} = \int f(x)\,dx$

Solve $\frac{dy}{dx} = (x^2+1)(y^2+1)$

$\Rightarrow \int \frac{dy}{(y^2+1)} = \int (x^2+1)\,dx$

$\Rightarrow \tan^{-1}y = \frac{x^3}{3} + x + C$

Type - 04

Solve $\frac{d^2y}{dx^2} = f(x)$ Let $p = \frac{dy}{dx}$

$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = f(x)$

$\Rightarrow \int \frac{dp}{dx} = \int f(x) \Rightarrow dp \Rightarrow \int dp = \int f(x)\,dx$

$\Rightarrow p = f_1(x) + C$

$\Rightarrow \frac{dy}{dx} = f_1(x) + C \Rightarrow \int dy = \int f_1(x) + C\,dx$

$\Rightarrow y = f_2(x) + Cx + K$

Solve $\frac{d^2y}{dx^2} = 12x^2 + 12x$ Let $p = \frac{dy}{dx}$

$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = 12x^2 + 12x$

$\Rightarrow \left(\frac{dp}{dx}\right) = 12x^2 + 12x \Rightarrow \int dp = \int 12x^2 + 12x\,dx$

$\Rightarrow p = \frac{12x^3}{3} + \frac{12x^2}{2} + C$

$\Rightarrow p = 4x^3 + 6x^2 + C$

$\Rightarrow \frac{dy}{dx} = 4x^3 + x^2 + C \Rightarrow \int dy = \int 4x^3 + x^2 + C$

$\Rightarrow y = \frac{4x^4}{4} + \frac{x^3}{3} + Cx + K$

$\Rightarrow y = x^4 + \frac{x^3}{3} + Cx + K$

Particular Solution :-

Solve $\frac{dy}{dx} = \cos x$, given that $y = 2$ when $x = 0$

$\Rightarrow \int dy = \int \cos x\,dx$

$\Rightarrow y = \sin x + C$

$\Rightarrow 2 = \sin 0 + C$

$\Rightarrow 2 = 0 + C$

$\Rightarrow C = 2$

Hence the particular solⁿ is $y = \sin x + 2$

Solve $\frac{d^2y}{dx^2} = 6x$ given that $y = 1$, $\frac{dy}{dx} = 2$ when $x = 0$

Let $\frac{dy}{dx} = p$

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 6x$

$\Rightarrow \frac{d}{dx}p = 6x \Rightarrow \int dp = \int 6x\,dx$

$\Rightarrow p = \frac{6x^2}{2} + C \Rightarrow p = 3x^2 + C$

$\Rightarrow \frac{dy}{dx} = 3x^2 + C \Rightarrow \int dy = \int 3x^2 + C\,dx$

$\Rightarrow y = \frac{3x^3}{3} + Cx + K$

$\Rightarrow y = x^3 + Cx + K$

$\Rightarrow 1 = 0 + 0 + 0 + K$

$\Rightarrow K = 1$

$\Rightarrow \frac{dy}{dx} = 3x^2 + C \Rightarrow 2 = 0 + C$

$\Rightarrow C = 2$

Hence particular solution $y = x^3 + 2x + 1$

Type - 05

Solve $\frac{dy}{dx} = f(ax + by + c)$

Q. Solve $\frac{dy}{dx} = \cos(x+y)$ Let $x + y = t$

$\Rightarrow \frac{dt}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \frac{dt}{dx} - 1 = \frac{dy}{dx}$

$\Rightarrow \frac{dt}{dx} - 1 = \cos t$

$\Rightarrow \frac{dt}{dx} = 1 + \cos t$

$\Rightarrow \int \frac{dt}{1+\cos t} = \int dx$

$\Rightarrow \int \frac{dt}{2\cos^2 \frac{t}{2}} = \int dx \Rightarrow \frac{1}{2}\int \sec^2 \frac{t}{2}\, dt = \int dx$

$\Rightarrow \frac{1}{2} \cdot \frac{\tan t/2}{1/2} = x + C$

$\Rightarrow \tan t/2 = x + C$

Type - 07 · Homogeneous Differential equation

solve $\boxed{\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}}$

Q. solve $(x+y)\, dy + (x-y)\, dx = 0$

$\Rightarrow (x+y)\, dy = -(x-y)\, dx$

$\Rightarrow \frac{dy}{dx} = \frac{y-x}{x+y}$ (Homogeneous)

let $y = vx$

$\Rightarrow \frac{d(vx)}{dx} = \frac{(vx) - x}{x + (vx)}$

$\Rightarrow x\frac{dv}{dx} + v = \frac{x[v-1]}{x[v+1]}$

$\Rightarrow x\frac{dv}{dx} = \frac{[v-1]}{[v+1]} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{v - 1 - v - v^2}{1+v}$

$\Rightarrow x\frac{dv}{dx} = \frac{-(1+v^2)}{1+v}$

$\Rightarrow \int \frac{1+v}{1+v^2}\, dv = -\int \frac{1}{x}\, dx$

$\Rightarrow \int \frac{1}{1+v^2}\, dv + \frac{1}{2}\int \frac{2v}{1+v^2}\, dv = -\int \frac{1}{x}\, dx$

$\Rightarrow \tan^{-1} v + \frac{1}{2}\ln|1+v^2| = -\ln x + C$

$\Rightarrow \tan^{-1}\frac{y}{x} + \frac{1}{2}\ln\left|1 + \frac{y^2}{x^2}\right| = -\ln x + C$

$\Rightarrow \tan^{-1}\frac{y}{x} + \ln\left(\frac{x^2+y^2}{x^2}\right)^{1/2} = -\ln x + C$

$\Rightarrow \tan^{-1}\frac{y}{x} + \ln\sqrt{y^2+x^2} - \ln x = -\ln x + C$

$\Rightarrow (x^2 - y^2)\, dx + 2xy\, dy = 0$

$\Rightarrow 2xy\, dy = -(x^2 - y^2)\, dx$

$\Rightarrow \frac{dy}{dx} = \frac{-x^2 - y^2}{2xy} \Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

$\Rightarrow \frac{d(vx)}{dx} = \frac{(vx)^2 - x^2}{2x(vx)}$ let $y = vx$

$\Rightarrow x\frac{dv}{dx} + v = \frac{v^2x^2 - x^2}{2x^2v}$

$\Rightarrow x\frac{dv}{dx} + v = \frac{x^2[v^2-1]}{x^2\, 2v}$

$\Rightarrow x\frac{dv}{dx} + v = \frac{[v^2-1]}{2v}$

$\Rightarrow x\frac{dv}{dx} = \left[\frac{v^2-1}{2v}\right] - v$

$\Rightarrow x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v} \Rightarrow x\frac{dv}{dx} = \frac{-v^2-1}{2v}$

$\Rightarrow x\frac{dv}{dx} = \frac{-(1+v^2)}{2v}$

$\Rightarrow \int \frac{2v}{1+v^2}\, dv = -\int \frac{1}{x}\, dx \Rightarrow \ln|1+v^2| = -\ln x + \ln$