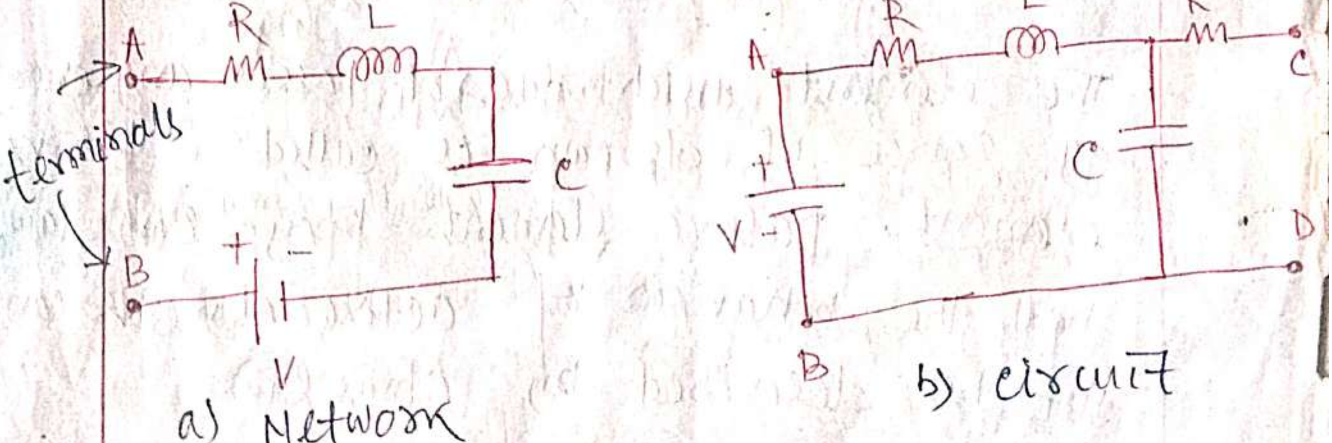
Unit-1: Circuit Elements & Energy Sources

1.1 Circuit Elements, scope of network analysis and synthesis.

Network and circuit: The interconnection of a number of circuit elements (i.e, voltage and current sources, resistor, inductor capacitor, diode, transistor etc) is called an electrical network.

If the network contains at least one closed path, then it is called an electric circuit.

Every circuit is a network but all networks are not circuits.



a) Network b) circuit

Circuit elements (or components)

Any circuit or network can be divided into verious regions or blocks. Each region has at least two accessible points called terminals. So a network (circuit) element or component is defined as a two terminal electrical device which can not be subdivided into further two terminal devices.

Ex: voltage source, current source, resistor, inductor, capacitor, diode, transistor, LED etc

Elements can be divided as —

a) active and passive elements

b) unilateral and bilateral elements

c) linear and non linear elements.

Active and passive elements:

An element which is a source of electrical energy or which is capable of increasing the level of signal energy is termed as an active element.

Ex: Batteries, BJTs, FETs, OP-Amps, etc

The element which does not posses any energy source of its own is called a passive element. passive elements obey's Ohm's law, but the behaviour of active elements can not be described by Ohm's law.

Ex: Resistor, inductor, capacitor, LDR, VDR etc.

Unilateral and Bilateral elements

If the magnitude of current flowing through a circuit element is affected when the polarity of the applied voltage is changed, the element is called unilateral element. Ex: diode, transistor etc.

If the magnitude of current flowing through the circuit element is not affected by changing the polarity of the applied voltage, the element is called bilateral element

Ex: Resistor.

Linear and non-linear elements:

If the element exhibits a linear relationship between applied voltage and produced current then the element is called linear element

Ex: Resistors.

If the element exhibits a nonlinear relationship between applied voltage and produced current, then the element is called non-linear element. Ex: voltage dependent resistor (VDR), light dependent ~~Resistor~~ Resistor (LDR) etc.

Network Analysis and Synthesis

Network (circuit) analysis is the process of determining the current that flows in one or more components of an electric circuit and/or the voltage that exists at various points in the circuit or the power delivered to (or consumed by) various components.

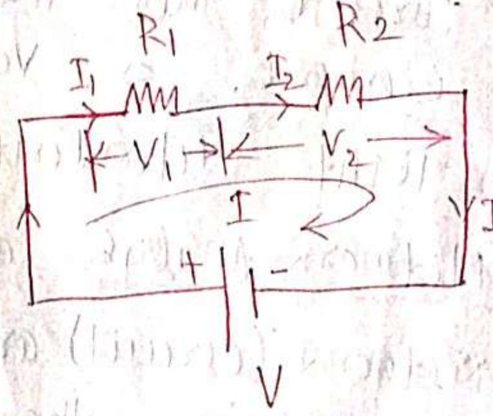
Network Synthesis is the process of selecting components having certain specified properties and interconnecting them

④ in a way necessary to achieve a desired goal.

1.2. Voltage Division, current division, energy sources

Voltage division: when a number of elements are connected in series across an energy (voltage or current) source, same current flows through each element but the total supply voltage get divided across the elements depending upon their individual resistance value. This process is called voltage division and it happens when the elements are connected in series.

Ex: in the fig. is shown two resistors R_1 and R_2 connected in series across a supply voltage of V volts.



current in the circuit I = current across $R_1 = I_1$ = current across $R_2 = I_2$

so, $I = I_1 = I_2$

But voltage across $R_1 = I_1 \cdot R_1 = I \cdot R_1$

and voltage across $R_2 = I_2 \cdot R_2 = I \cdot R_2$

Total supply voltage $V = V_1 + V_2 = IR_1 + IR_2$

$= I(R_1+R_2) = I.R$ where R is the combined resistance of R_1 and R_2 connected in series

i.e., $R = R_1 + R_2$.

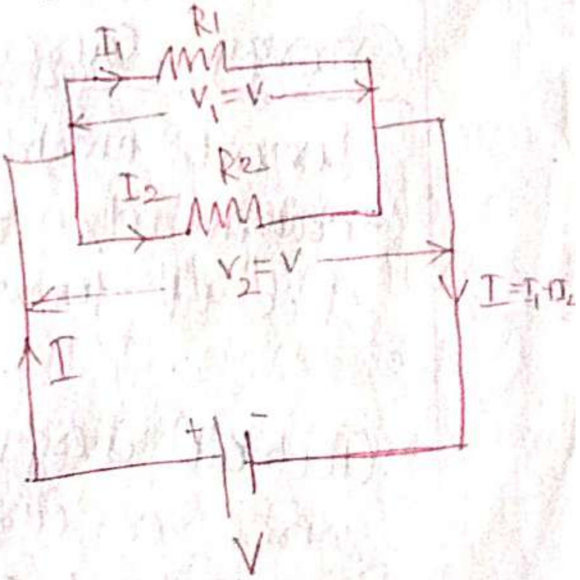
So the total voltage is divided across R_1 and R_2.

$\therefore V_1 = I \cdot R_1 = \frac{V}{R_1+R_2} \cdot R_1$ or $\boxed{V_1 = \frac{R_1}{R_1+R_2} \cdot V}$

and $V_2 = I \cdot R_2 = \frac{V}{R_1+R_2} \cdot R_2$ or, $\boxed{V_2 = \frac{R_2}{R_1+R_2} \cdot V}$

Current division: when a number of elements are connected in parallel across an energy source, same voltage will be developed across each element but the current through the elements will be the division of total current supplied by the source, depending upon the resistance values of the elements.

In the fig. is shown two resistors R_1 & R_2 are connected in parallel across a voltage source of V volts.



Now voltage across R_1

= voltage across R_2

= supply voltage i.e, $V = V_1 = V_2$

$= I_1R_1 = I_2R_2$

But current across R_1 (i.e, I_1) $\neq$ current across R_2 (i.e, I_2)

But total current $I = I_1 + I_2$ i.e, sum of two currents

(6) If R is the combined resistance of R_1 and R_2 connected in parallel then $R = \frac{R_1 \cdot R_2}{R_1 + R_2}$

and total current $I = \frac{V}{R} = \frac{V(R_1 + R_2)}{R_1 \cdot R_2}$.

$$\text{or, } V = I \cdot R = \frac{I \cdot R_1 \cdot R_2}{R_1 + R_2}$$

Now current in R_1 i.e, $= I_1 = \frac{V_1}{R_1} = \frac{V}{R_1}$

$$= \frac{1}{\cancel{R_1}} \times \frac{I \cdot \cancel{R_1} \cdot R_2}{R_1 + R_2} = \frac{R_2}{R_1 + R_2} \times I.$$

$$\text{or, } \boxed{I_1 = I \cdot \frac{R_2}{R_1 + R_2}}$$

Similarly current in R_2 {i.e, I_2} $= \frac{V_2}{R_2} = \frac{V}{R_2} = \frac{1}{R_2} \times \frac{I \cdot R_1 \cdot R_2}{R_1 + R_2}$

$$\text{or, } \boxed{I_2 = I \cdot \frac{R_1}{R_1 + R_2}}$$

Energy sources: Source is a basic network element which supplies energy to a network through its terminals.

It is of two types — a) independent sources
b) dependent source.

Electrical energy sources can be either voltage source or current source. Therefore independent ~~of~~ electrical energy sources can be independent voltage source or independent current source.

independent voltage source. It is a two terminal network element that establishes a specified voltage across the terminals. The terminal voltage may be a constant voltage (dc voltage in fig ⓐ) or a specified function of time t (ac voltage as in fig ⓑ)

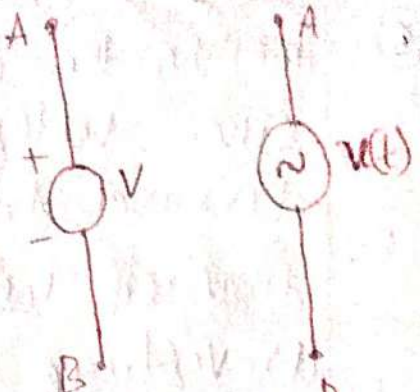


fig ⓐ fig ⓑ
independent voltage sources

Independent current source:

It is a two terminal network element that produces a specified current through its terminals. The terminal current may be constant or a specified function of time.

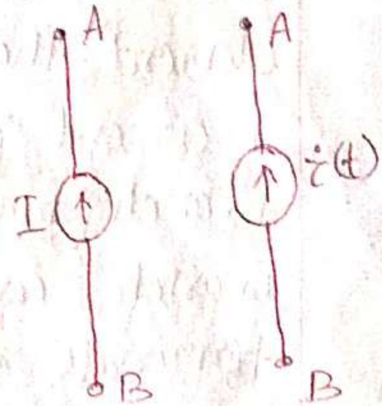


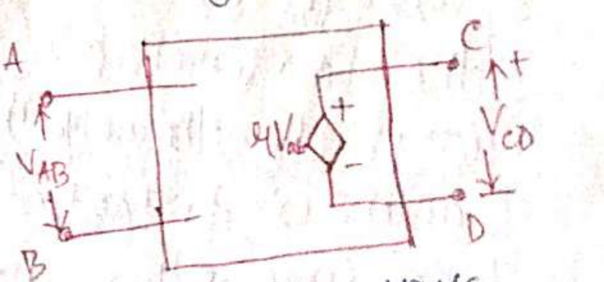
fig ⓐ fig ⓑ
independent current sources.

Dependent sources:

If the voltage or current of a source is dependent upon some other voltage or current then it is called a dependent source. It is of 4-types, depending upon whether the controlled variable is voltage or current and the controlled source is a voltage source or current source.

a) voltage controlled (dependent) voltage source (VCVS or VDV

It is a four terminal network element in which the voltage V_{CD} between the two terminals C & D,



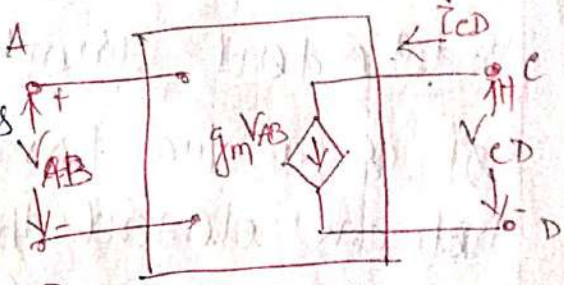
VCVS or VDVS

(8)

(may be the output terminals) depends upon some other voltage V_{AB}, between any two other terminal A, B. so $\boxed{V_{CD} = \mu V_{AB}}$ where μ is the dimension less constant called voltage gain.

(b) voltage controlled (dependent) current source. (VCCS or VDCS)

It is a four terminal network element that establishes a current i_{CD} in a branch (may be the branch connecting output terminals C, D) which is proportional to the voltage V_{AB} across any other terminals A, B (may be the input terminals)

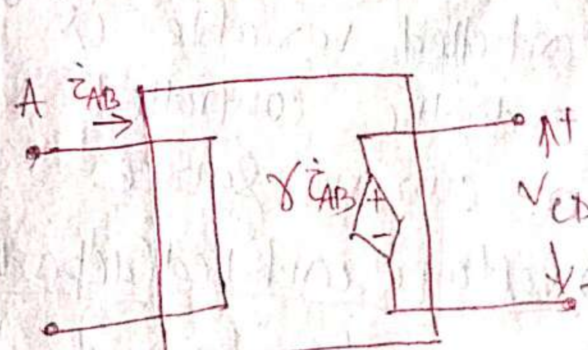


VCCS

so, $\boxed{i_{CD} = g_m V_{AB}}$ where $g_m \Rightarrow$ is called the trans or mutual conductance with unit amp/volt or (siemens)

(c) current controlled (dependent) voltage source. (CCVS or CDVS)

It is a four terminal network element, that establishes a voltage V_{CD} across the terminals C, D (may be the output)



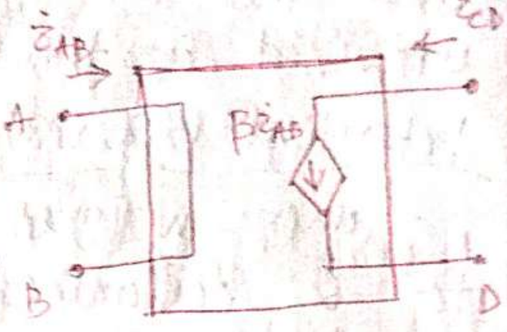
(CCVS)

which is proportional to the current i_{AB} flowing in some other branch A, B (may be the inputs

Hence $\boxed{v_{CD} = r\, i_{AB}}$ where r is the transfer or mutual resistance of unit volt/amp or ohm (Ω)

(d) current controlled (dependent) current source: (CCCS or CDCS)

It is a four terminal network element which establishes a current i_{CD} in a branch CD (may be on output side) which is proportional to the current i_{AB} in some other branch AB. Here



(CCCS)

$$\boxed{i_{CD} = \beta\, i_{AB}}$$

where β is a dimensionless constant called current gain.

1.3 Electric charge, current, energy, potential, R-L-C parameters

Electric charge comes from the concept of atomic structure, where atom consists of nucleus and extranuclear part. Nucleus consists of proton and neutron and in extranuclear part, electrons revolve round the nucleus. protons are +vely charged bodies and electrons are -vely charged bodies.

when electrons comes out of the atomic bond (by applying external energy in the form of heat, light, electricity etc), they flow as -ve charge carriers producing current and at the same time +vely charged ions are also produced.

unit of electric charge is coulomb (coul)
and unit of electric current is coul/sec or Ampere

like charges repel eachother and unlike charges attract. To overcome this force of attraction (or

(10) repulsion) an electromotive force must be applied. When the charges are separated, it is said that a potential difference (which is the difference in electric potential energy of the charges) exists and the work or energy per unit charge utilized in this process is called <u>voltage or potential difference</u>.

When W joules of energy (work) is supplied to Q coulomb of charge, the voltage is given by – $\boxed{V = \frac{W}{Q}}$ volts or $\boxed{\text{volt} = \frac{\text{Joul}}{\text{coul}}}$

The phenomena of transfer of charge from one point to another is termed as <u>electric current</u>. current (I) is defined as the rate of flow of electrons (charges) in a conductor and is measured by the number of electrons (quantity of charge) that flow in unit time. $\boxed{I = \frac{Q}{t}}$ ampere, $\boxed{\text{Ampere} = \frac{\text{Culomb}}{\text{second}}}$

<u>Electrical energy</u> is the total workdone in the electric circuit. The rate at which the work done in an electric circuit is called <u>electric power</u>.

$$\text{power} = \frac{\text{workdone}}{\text{time}} = \frac{\text{Energy}}{\text{time}}, \quad \boxed{P = \frac{W}{t}} \text{ watt or } \boxed{\text{watt} = \frac{\text{Joule}}{\text{second}}}$$

$$\text{Now } P = \frac{W}{t} = \frac{W}{Q} \cdot \frac{Q}{t} = V \cdot I \text{ or } \boxed{P = VI} \quad \boxed{\text{Watt} = \text{volt} \times \text{Amp}}$$

Electrical energy is measured in Joules where as electric power is measured in watts.

R-L-C (Resistance- Inductance- Capacitance) parameter

a) **Resistance**: Resistance is defined as the property of a material which opposes the flow of current through it. unit of resistance is ohm(Ω)

From ohm's law, resistance $= \frac{\text{voltage}}{\text{current}}$

or, $\boxed{\text{Ohm} = \frac{\text{Volt}}{\text{Ampere}}}$

If one ampere current flows through a conductor by applying one volt of potential difference between the end of the conductor then the resistance offered by a conductor is said to be one ohm ($1\,\Omega$).

Resistance of a conductor is —

- directly proportional to the length of the conductor
- inversely proportional to the area of cross section of the conductor
- depends upon nature of material
- depends upon temperature of the conductor.

$R \propto L$, $R \propto \frac{1}{A}$ or $R \propto \frac{L}{A} \Rightarrow \boxed{R = \rho \cdot \frac{L}{A}}$

where ρ is the proportionality constant, representing nature of the material and is called specific resistance or resistivity.

so, $\boxed{\rho = \frac{RA}{L}}$ $\quad \frac{\text{ohm} \cdot m^2}{m}$ or ohm-m.

voltage-current relation in a resistor is

$\boxed{V = R \cdot I}$ or $\boxed{I = \frac{V}{R}}$

(12)

Power dissipated in a resistor:

When current flows through a resistor, power is absorbed by the resistor, which is given by $P = Vi = R \cdot i \cdot i = Ri^2$ watt.

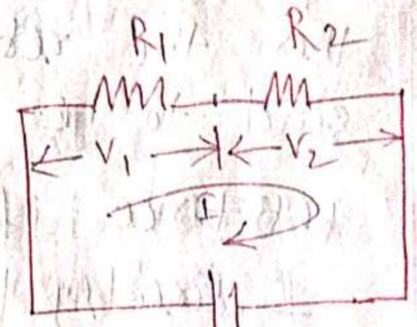
Power dissipated in the resistor, is produced in the form of heat (thermal energy), which can be represented as —

$$W = \int_0^t V i \, dt = \int_0^t R i^2 \cdot dt = i^2 \cdot R \cdot t$$

$$\therefore \boxed{W = i^2 R t} \text{ joules}$$

Resistor connections

i) Resistors in series:



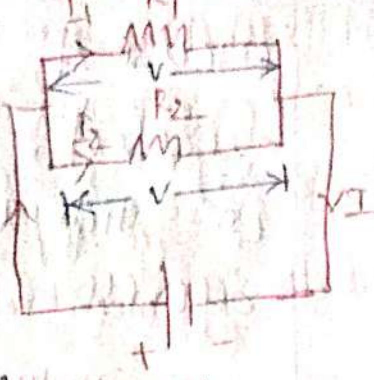
A number of resistors are said to be connected in series when they are connected end to end so that same current flows through each and the ~~appl~~ applied voltage is sum of voltages developed across the resistors.

If V_1 is the voltage across R_1 and V_2 is the voltage across R_2 and if R is the combined resistance of series connection of R_1 & R_2, then $V = V_1 + V_2$ or $IR = IR_1 + IR_2$

or $\boxed{R = R_1 + R_2}$

So resistances are added up when connected in series.

(13) ii) Resistors in parallel



A number of resistances are said to be connected in parallel ~~series~~ when one end of each resistance combiningly connected to one terminal of the battery and other end of each resistance connected to other end of the battery so that same voltage developed across each resistor (equal to supply voltage) and ~~total~~ current supplied by the battery is sum of currents flowing through the resistors.

If I_1 is the current in R_1 and I_2 is that in R_2 and R is the combined resistance of R_1 in parallel with R_2, then —

$$I = I_1 + I_2 \quad \text{or} \quad \frac{V}{R} = \frac{V}{R_1} + \frac{V}{R_2}$$

$$\text{or} \quad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1 \cdot R_2}$$

$$\text{or,} \quad \boxed{R = \frac{R_1 R_2}{R_1 + R_2}}$$

(b) Inductance (L)

Inductance is defined as the property of a coil which opposes the change of current through the coil.

If current in the coil is increased, then self induced emf is produced in the coil in such a ~~way~~ direction that it opposes the increse of current. Similarly when current is decresed, the self induced emf is produced in such a direction that it will oppose the decrese in current.

Inductance is defined as the ratio of flux linkage to current flowing in the coil. unit of inductance is Henry (H)

$$\boxed{L = \frac{N\phi}{I}} \quad \boxed{\text{Henry} = \frac{\text{weber-turns}}{\text{ampere}}}$$

A coil is said to have an inductance of 1 Henry, if a current of 1 Ampere flowing through it produces a flux linkage of 1 weber-turn in it.

Inductance of an inductor is —
- directly proportional to square of no. of turns
- directly proportional to the area of cross section of the coil.
- inversely proportional to the length
- it depends upon the nature of the magnetic material (represented by permiability

$L \propto N^2$, $L \propto A$ and $L \propto \frac{1}{l}$

combiningly $L \propto \frac{N^2 A}{l}$ or $\boxed{L = \mu \frac{N^2 A}{l}}$

where μ is called the absolute permiability of the coil and is given by $\boxed{\mu = \frac{L \cdot l}{N^2 A}}$

unit of μ is Henry/meter.

(11) voltage current relationship in an inductor is given by $V = L\frac{di}{dt}$ where v is the applied voltage and i is the current in the inductor of self inductance L Henry.

as $V = L\frac{di}{dt}$ or $di = \frac{1}{L} V\,dt$

now integrating both sides - $\int_{i(0)}^{i(t)} di = \frac{1}{L}\int_0^t V\,dt$

or $\boxed{i(t) = \frac{1}{L}\int_0^t V\,dt + i(0)}$ where $i(0)$ is the initial current

if $i(0) = 0$, then $\boxed{i(t) = \frac{1}{L}\int_0^t V\,dt}$

Energy stored in an inductor

in an inductor $v = L\,di/dt$

Now energy supplied to the inductor during time interval dt is given by -

$$dw = v\,i\,dt = L\frac{di}{dt}\,i\,dt = L\,i\,di$$

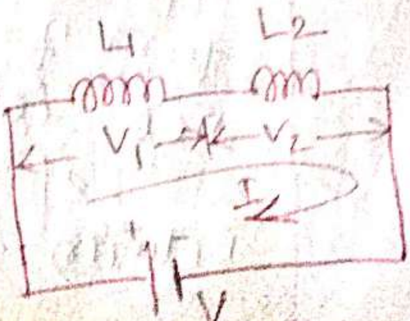
Hence total energy supplied to the inductor when the current is increased from '0' to 'I' ampere, can be obtained by integrating both sides of the above eqn -

$$\therefore W = \int_0^I L\,i\,di = \frac{1}{2}LI^2$$

$\therefore \boxed{W = \frac{1}{2}LI^2}$ is the energy stored in an inductor.

inductance connections

i) inductances in series:



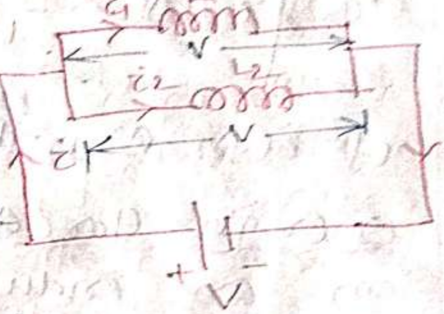
inductors are said to be connected in series

(16) when same same current flows through each and voltage is additive. If V_1 is the voltage across L_1 and V_2 is that in L_2 and L is the combined inductance of L_1 in series with L_2, then $V = V_1 + V_2$

or $L \cdot \frac{di}{dt} = L_1 \frac{di}{dt} + L_2 \frac{di}{dt}$

or $\boxed{L = L_1 + L_2}$ so inductances get added when connected in series.

inductances in parallel



inductances are said to be connected in parallel when same voltage drops across each and current becomes additive.

if i_1 is current in L_1 and i_2 is that in L_2 and L is the combined inductance of L_1 in parallel with L_2, then —

$i = i_1 + i_2$ or $\frac{1}{L} \int V \cdot dt = \frac{1}{L_1} \int V dt + \frac{1}{L_2} \int V dt$

or $\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} = \frac{L_1 + L_2}{L_1 \cdot L_2}$ or $\boxed{L = \frac{L_1 \cdot L_2}{L_1 + L_2}}$

(c) capacitance (C)

capacitance is defined as the capacity of a capacitor to store electric charge (in the form of voltage) when a potential difference is applied between the plates.

If Q coulomb charge is stored between the plates by applying a potential

(17) difference of V volts, then capacitance is given by

$$\boxed{C = \frac{Q}{V}} \qquad \boxed{\text{Farad} = \frac{\text{Coul}}{\text{volt}}}$$

One farad is the capacitance obtained by storing a charge of one coulomb when the potential difference maintained between the plates is one volt.

Capacitance of a capacitor is —
- directly proportional to the area of the plates
- inversely to the distance between the plates.
- depends upon the nature of the medium between the plates (represented by permittivity)

$C \propto A$; $C \propto \frac{1}{d}$ or $C \propto \frac{A}{d}$ or $\boxed{C = \epsilon \frac{A}{d}}$

where ϵ is the absolute permittivity of the medium between the plates, and is given by $\boxed{\epsilon = \frac{C \cdot d}{A}}$ unit of ϵ is farad/meter (F/m)

<u>Current voltage relationship in a capacitor</u>

Now $q = C V$ or $\frac{dq}{dt} = C \cdot \frac{dv}{dt} = i$

$\therefore \boxed{i = C \frac{dv}{dt}}$

Similarly $i\,dt = C\,dv$ or $dv = \frac{1}{C} i\,dt$

or, $\int_{V(0)}^{V(t)} dv = \frac{1}{C}\int_0^t i \cdot dt$ or, $\boxed{v(t) = \frac{1}{C}\int_0^t i\,dt + V(0)}$

where $V(0)$ is the initial voltage stored in the capacitor, if $V(0) = 0$, then $\boxed{V(t) = \frac{1}{C}\int_0^t i \cdot dt}$

(18)

Energy stored in a capacitor:

When a potential difference of v volts applied to a capacitor of 'C' farad, then current in the capacitor is given by —

$i = C\frac{dv}{dt}$. Energy supplied to the capacitor during time interval 'dt' is given by

$$dw = v \cdot i \cdot dt = v \cdot \frac{C \cdot dv}{dt} \cdot dt = v \cdot C \cdot dv$$

Hence total energy supplied to the capacitor when potential difference is increased from '0' to 'v' volts is given by —

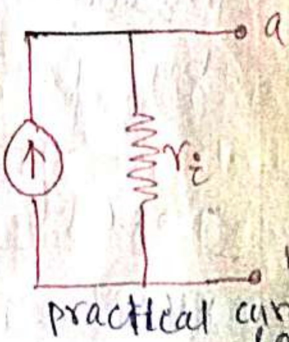
$$W = \int_0^v C \cdot v \cdot dv = \frac{1}{2} C v^2$$

$\therefore$ $\boxed{W = \frac{1}{2} C v^2}$ Joule, is the energy stored in a capacitor.

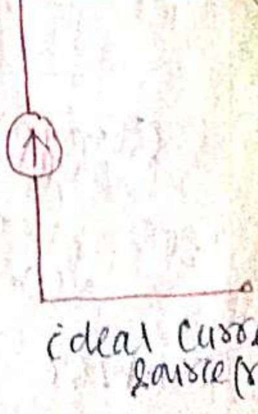
(*) 1.4. Capacitor connections: (Page - 21)

Current & voltage sources and their transformation, mutual inductance

A practical current source is always represented with its internal resistance in parallel with it. For an ideal current source internal resistance is very very large ($r_i \simeq \infty$) and hence it is represented by an open circuit as shown in the figure.

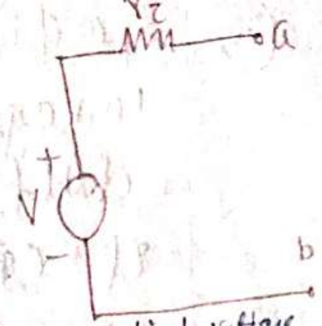


practical current source

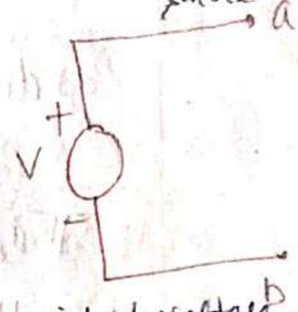


ideal current source ($r_i \simeq \infty$)

Similarly a practical voltage source is always represented with its internal resistance in series with it. For an ideal voltage source, the internal resistance is very very less ($r_i \simeq 0$) and hence it is represented by a short circuit as shown in the figure.



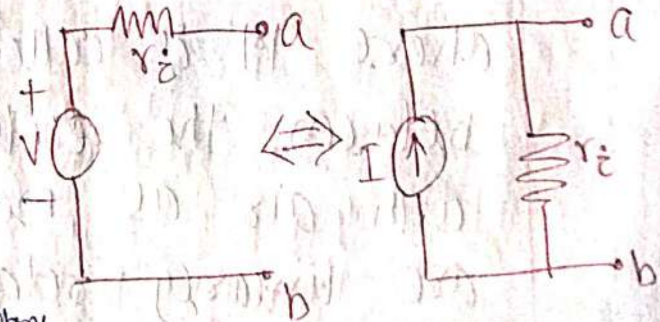
practical voltage source.



ideal voltage source ($r_i \simeq 0$)

A voltage source in series with its internal resistance can be transformed to a current source in parallel with the same internal resistance and vice-versa.

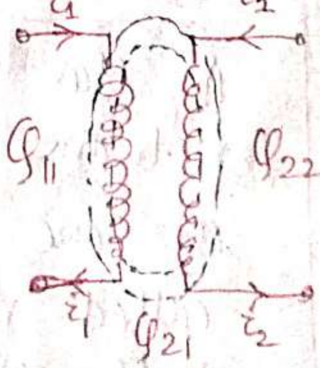
If the voltage source has value 'V' volts in series with internal resistance r_i ohms, then its transformed current source will have current $\boxed{I = \frac{V}{r_i}}$



Similarly a current source of I Amp with internal resistance r_i ohms can be transformed to a voltage source of $\boxed{V = I \cdot r_i}$

Mutual Inductance:

Let take two closely spaced coils having turns N_1 & N_2, carrying current



i_1 and i_2 respectively. ϕ_{11} and ϕ_{22} are flux linkages of coil 1 with itself and of coil 2 with itself respectively. Let ϕ_{12} is the flux linkage of coil 1 with coil 2 and ϕ_{21} is the flux linkage of coil 2 with coil 1.

Induced voltage in coil 2 is given by

$$V_{L2} = N_2 \frac{d\phi_{12}}{dt} \quad \text{——} (1)$$

This induced voltage is also proportional to the rate of change of i_1, so

$$V_{L2} = M \frac{di_1}{dt} \quad \text{——} (2)$$

Now equating (1) & (2)

$$M \frac{di_1}{dt} = N_2 \frac{d\phi_{12}}{dt} \quad \therefore \boxed{M = N_2 \frac{d\phi_{12}}{dt}}$$

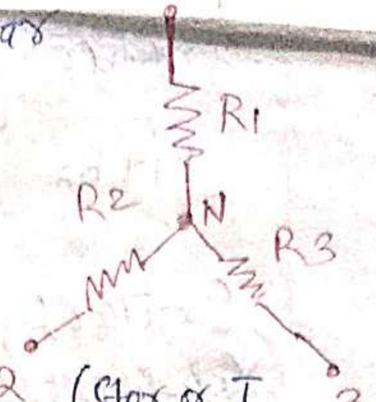
Similarly it can be stated that –

$$\boxed{M = N_1 \frac{d\phi_{21}}{dt}}$$

where M is called the mutual inductance between the coils. If the coils are linked with air as medium, the flux and currents are linearly related and the expression for mutual inductance will be $\boxed{M = N_2 \frac{\phi_{12}}{i_1} = N_1 \frac{\phi_{21}}{i_2}}$

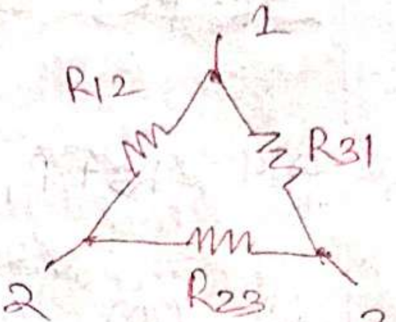
1.5 Star - Delta transformation:

A star network is a special type of network having 3 terminals connected at one end to a common point (node) called neutral point and 3 other terminals connected to the 3-phases of a 3ϕ power supply. It looks like a star or (english letter T) and so called as star or T network.



(Star or T network)

Similarly a delta network is a special type of network in the form of triangular closed loop, having 3 resistances in the 3 arms and 3 terminals are taken from the vertices.



(Delta or π network)

It looks like Δ (delta) or π (pie) and so called delta or pie network.

a) Delta to star conversion.

In fig ① and ② are shown delta and star networks along with resistances as shown.

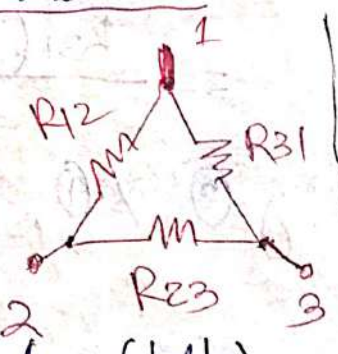


fig 1 (delta)

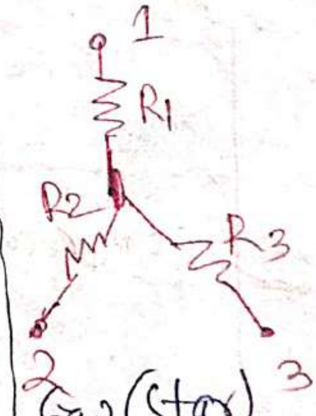


fig 2 (star)

Resistances in the delta network are given to us and we have to find the equivalent star network values.

From fig ①, equivalent resistance between terminals 1 and 2 is $R_{12} \parallel (R_{23}+R_{31}) = \dfrac{R_{12}(R_{23}+R_{31})}{R_{12}+R_{23}+R_{31}}$

From fig ② equivalent resistance between terminals 1 and 2 is $R_1 + R_2$

considering both networks are equivalent -

$$R_1 + R_2 = \frac{R_{12}(R_{23} + R_{31})}{R_{12} + R_{23} + R_{31}} \quad \text{——— (1)}$$

Similarly considering equivalent resistances between terminals 2 and 3 from both figures -

$$R_2 + R_3 = \frac{R_{23}(R_{31} + R_{12})}{R_{12} + R_{23} + R_{31}} \quad \text{——— (2)}$$

and $$R_3 + R_1 = \frac{R_{31}(R_{12} + R_{23})}{R_{12} + R_{23} + R_{31}} \quad \text{——— (3)}$$

we can solve the above 3 equations to find the values of R_1, R_2, R_3 in terms of R_{12}, R_{23} & R_{31}

Subtracting (2) from (1) —

$$R_1 - R_3 = \frac{R_{12}(R_{23} + R_{31}) - R_{23}(R_{31} + R_{12})}{R_{12} + R_{23} + R_{31}}$$

$$= \frac{R_{31}(R_{12} - R_{23})}{\Sigma R_{12}} \quad \text{— (4)}$$ put $R_{12} + R_{23} + R_{31} = \Sigma R_{12}$

Comparing (3) & (4), $$R_1 = \frac{R_{31}(R_{12} + R_{23}) + R_{31}(R_{12} - R_{23})}{2\,\Sigma R_{12}}$$

$$= \frac{2\,R_{31}R_{12}}{2\,\Sigma R_{12}} = \frac{R_{12}\cdot R_{31}}{\Sigma R_{12}}$$

and $$R_3 = \frac{R_{31}(R_{12} + R_{23}) - R_{31}(R_{12} - R_{23})}{2\,\Sigma R_{12}}$$

$$= \frac{2\,R_{31}\cdot R_{23}}{2\,\Sigma R_{12}} = \frac{R_{31}\cdot R_{23}}{\Sigma R_{12}}$$

$$\therefore \boxed{R_1 = \frac{R_{12}\cdot R_{13}}{\Sigma R_{12}}} \quad \boxed{R_3 = \frac{R_{31}\cdot R_{32}}{\Sigma R_{12}}}$$

Putting the value of R_3 in eqn ②.

$$R_2 = \frac{R_{23}(R_{31}+R_{12})}{\Sigma R_{12}} - \frac{R_{31}\cdot R_{23}}{\Sigma R_{12}}$$

$$= \frac{R_{23}\cdot R_{31} + R_{23}\cdot R_{12} - R_{31}\cdot R_{23}}{\Sigma R_{12}} = \frac{R_{23}\cdot R_{12}}{\Sigma R_{12}}$$

or $\boxed{R_2 = \frac{R_{21}\cdot R_{23}}{\Sigma R_{12}}}$ So from the given delta values (R_{12}, R_{23} & R_{31}) we will get star values R_1, R_2, R_3.

b) Star to Delta conversion:

From the values of R_1, R_2, R_3 in terms of R_{12}, R_{23} and R_{31}, multiplying pairs of star resistances—

$$R_1\cdot R_2 = \frac{R_{12}^2\cdot R_{23}\cdot R_{31}}{(\Sigma R_{12})^2} \quad — ⑤$$

Similarly $R_2\cdot R_3 = \frac{R_{23}^2\cdot R_{31}\cdot R_{12}}{(\Sigma R_{12})^2} \quad — ⑥$

and $R_3\cdot R_1 = \frac{R_{31}^2\cdot R_{12}\cdot R_{23}}{(\Sigma R_{12})^2} \quad — ⑦$

Adding eqns ⑤, ⑥ and ⑦ —

$$R_1\cdot R_2 + R_2\cdot R_3 + R_3\cdot R_1 = \frac{R_{12}\cdot R_{23}\cdot R_{31}(R_{12}+R_{23}+R_{31})}{(\Sigma R_{12})^2}$$

$$= \frac{R_{12}\cdot R_{23}\cdot R_{31}}{R_{12}+R_{23}+R_{31}} = \frac{R_{12}\cdot R_{23}\cdot R_{31}}{\Sigma R_{12}}$$

$$= R_1\cdot R_{23} = R_2\cdot R_{31} = R_3\cdot R_{12}.$$

From the above, $R_{12} = \frac{R_1 R_2 + R_2\cdot R_3 + R_3\cdot R_1}{R_3}$

$$= \frac{R_1\cdot R_2}{R_3} + \frac{R_2\cdot \cancel{R_3}}{\cancel{R_3}} + \frac{\cancel{R_3}\cdot R_1}{\cancel{R_3}}$$

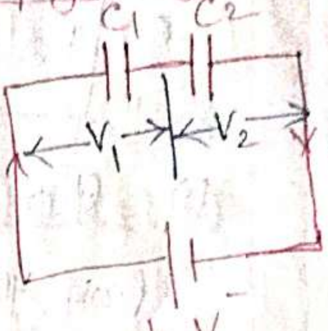
$$= R_1 + R_2 + \frac{R_1\cdot R_2}{R_3}$$

$\therefore \boxed{R_{12} = R_1 + R_2 + \frac{R_1 \cdot R_2}{R_3}}$ and $\boxed{R_{23} = R_2 + R_3 + \frac{R_2 \cdot R_3}{R_1}}$

and $\boxed{R_{31} = R_3 + R_1 + \frac{R_3 \cdot R_1}{R_2}}$ So we get delta resistances (R_{12}, R_{23}, R_{31}) from star values R_1, R_2, R_3.

Capacitor connections (from page 18)

Series connection of capacitors:



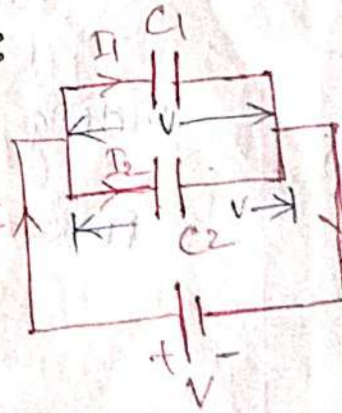
No. of capacitors are said to be series when same current flows through each and voltage get additive. i.e, $V = V_1 + V_2$

If C is the combined capacitance of the capacitors C_1 and C_2 connected in series, then—

$$\frac{1}{C}\int_0^t I \cdot dt = \frac{1}{C_1}\int_0^t I\,dt + \frac{1}{C_2}\int_0^t I\,dt \quad [\text{as } V = V_1 + V_2]$$

or, $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{C_1 + C_2}{C_1 \cdot C_2}$ or, $\boxed{C = \frac{C_1 \cdot C_2}{C_1 + C_2}}$

Parallel connection of capacitors:



No. of capacitors are said to be connected in parallel when same voltage is developed across each and current get additive. i.e, $I = I_1 + I_2$

If C is the equivalent capacitance of C_1 & C_2 connected in parallel then—

$I = I_1 + I_2$ or $C\frac{dV}{dt} = C_1\frac{dV}{dt} + C_2\frac{dV}{dt}$

or $\boxed{C = C_1 + C_2}$

— x —

Unit 2 Network Theorems

2.1. Nodal and Mesh analysis.

Nodal analysis: Analysis of a network or circuit, by considering the node voltage as the ~~variable~~ main or independent parameter and finding out other parameters from these values, is called nodal analysis.

Node (junction point): A node is defined as the meeting point of two or more branches. Each branch must contain at least one element and/or an energy source.

Nodal Analysis is based upon Kirchoff's current law (KCL) which is defined as the algebraic sum of currents meeting at a point (called node or junction) is zero. $\boxed{\Sigma I = 0}$ at a node.

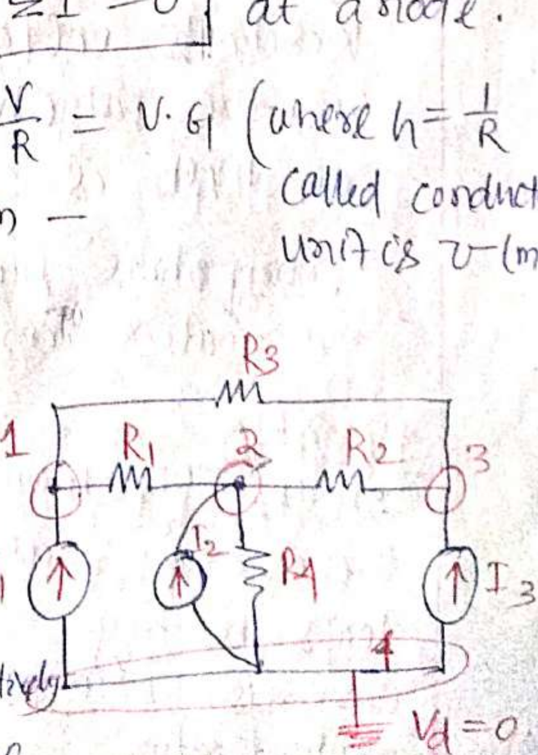
From Ohm's law $I = \frac{V}{R} = V \cdot G$ (where $G = \frac{1}{R}$ called conductance, unit is ℧ (mho))

writing in matrix form –

$$\boxed{[G][V] = [I]}$$

Example

The circuit shown in figure contains 3 distinct nodes 1, 2 & 3, having voltages V_1, V_2, & V_3 respectively. The fourth node 4 is the



datum or reference node having potential $V_4 = 0$. If the 3 current sources connected to nodes 1, 2, 3 are given to be I_1, I_2 & I_3 then the node voltages V_1, V_2, V_3 are related to them in the matrix eqn. as — $[G][V] = [I]$

in expand form —
$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

the values of V_1, V_2, V_3 can be found from crammer's rule.

<u>Mesh Analysis</u>: Analysis of a network or circuit by considering the loop currents as the independent parameter and finding out other parameters from it, is called mesh or loop analysis.

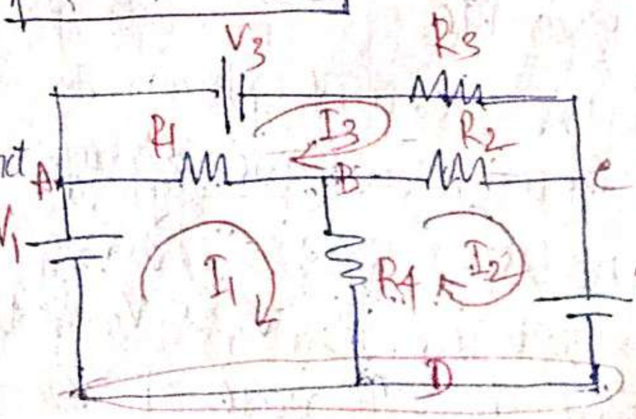
<u>Mesh or Loop</u>: is a closed path in a network or circuit. Mesh analysis is based upon Kirchoff's voltage law (KVL), which is defined as the algebraic sum of voltages in a loop or mesh is zero. $\boxed{\Sigma V = 0}$ in a loop or mesh.

from ohm's law — $V = I \cdot R$

In matrix form $\boxed{[R][I] = [V]}$

<u>Example</u>

In the fig shown — there are 3 distinct loops having currents I_1, I_2, I_3 respectively.



(3) writting KVL equations for the 3 loops and arranging them in matrix form — $[R][I] = [V]$, we will have

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

From the above matrix eqn, the values of I_1, I_2, I_3 can be obtained by crammers rule.

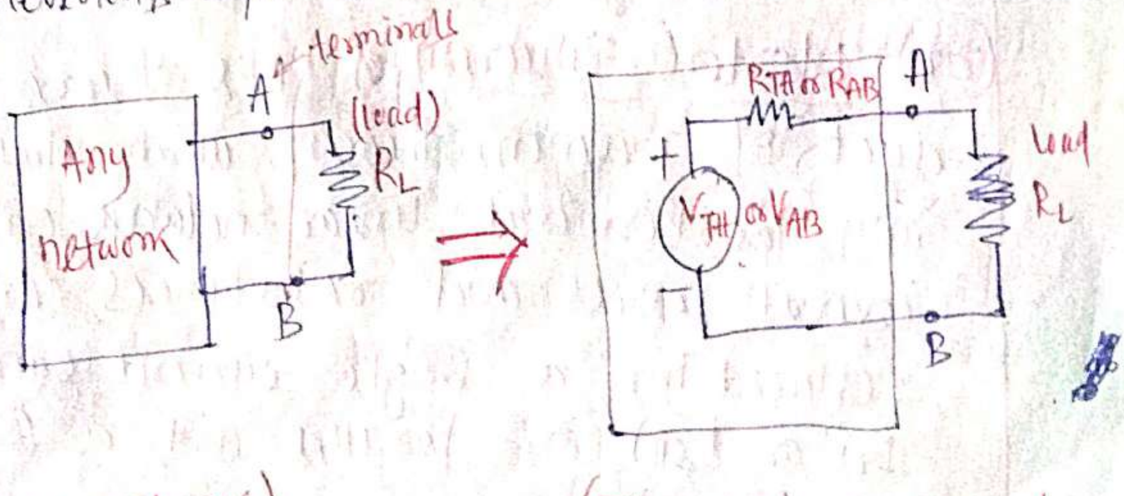
2.2. Network Theorems:

(i) Thevenin's Theorem: The theorem states that any two terminal linear network containing several sources and impedances (or resistances) can be replaced by a single voltage source in series with a single impedance (or resistance)

The voltage source (called Thevenin's voltage source of value V_{TH}) will have a value equal to the open circuit voltage across the terminals A & B (i.e, $V_{TH} = V_{AB}$)

The resistance (or impedance) is the resistance between terminals A & B, looking towards the network, when all the energy sources are replaced by their respective internal resistances.

Thevenin's equivalent resistance $Z_{TH} = Z_{AB}$.



(any network)

(Thevenin's equivalent network)

(4) Steps to be followed while aplying Thevenin's Theorem.

Step I: Remove the load resistance R_L and mark the terminals A, B.

Step-II: Find open circuit voltage across A & B by any method (KVL, KCL analysis or anyother) Now $V_{TH} = V_{AB}$.

Step-III Find $R_{TH} = R_{AB}$ which is the total resistance across A & B looking towards the network while replacing voltage source by short circuit (internal resistance of volt. source is zero ohm) and current source by open circuit (internal resistance of current source is infinity ohm).

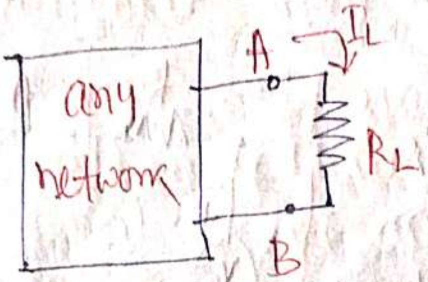
Step IV Draw Thevenin's equivalent circuit i.e, V_{TH} in series with R_{TH} and mark the terminals A, B. Now connect load resistance R_L across A & B.

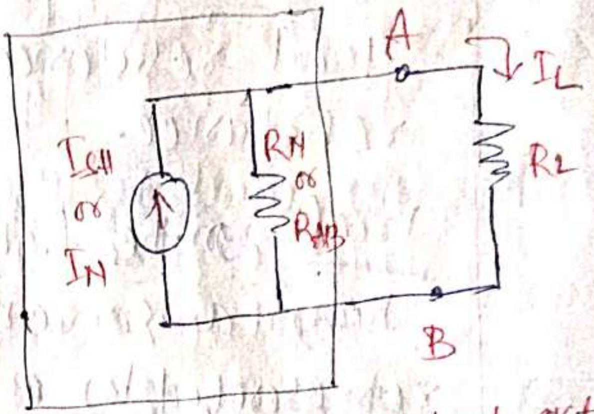
Step V: Find current in R_L (say) $I_L = \frac{V_{TH}}{R_{TH}+R_L}$

voltage across R_L (say) $V_L = I_L R_L = \frac{V_{TH}}{R_{TH}+R_L} \times R_L$

(ii) Norton's Theorem: This theorem is the dual of Thevenin's theorem, which states that - any two terminal linear network containing several sources and resistances can be replaced by a single current source (of value I_N or I_{SH}) in parallel with a single resistance (R_{AB}).

The single current source (called Norton's current source) has the value I_{SH} or I_N which is the short circuit current I_{AB} across the short circuited terminal A, B, and the parallel resistance R_N or R_{SH} is the open circuit resistance looking into the network across the terminals A, B, while all the sources are replaced by their respective internal impedances.





Norton's equivalent ckt.

Steps to be followed on applying Norton's Theorem

Step-I: Remove the load resistance R_L and mark the terminals A, B.

Step-II: Short the terminals A, B. Find current flowing from A to B. i.e., $I_{SH} = I_N = I_{AB}$.

Step-III: Find $R_{AB} = R_{TH}$, which is the open circuit resistance looking into the network through A, B while replacing sources by their respective internal impedances.

Step-IV: Replace the network by the Norton's equivalent ckt i.e, current source I_{SH} in parallel with R_{TH}.

Step-V: Connect the load resistance R_L across terminals A, B.

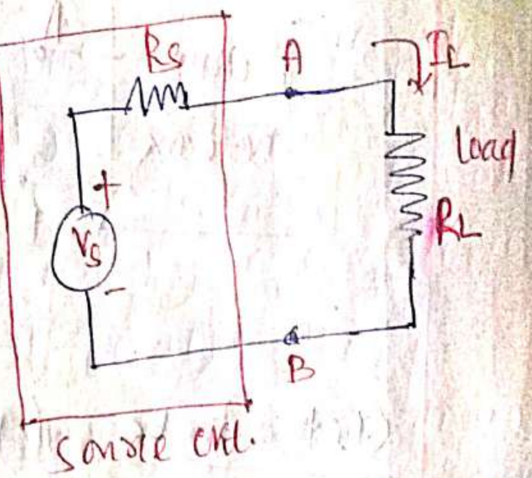
(6) Now Load current $I_L = I_{SH} \times \frac{R_{TH}}{R_{TH}+R_L}$

and voltage drop across load

$$V_L = I_L \times R_L = I_{SH} \times \frac{R_{TH}}{R_{TH}+R_L} \times R_L$$

(iii) Maximum power Transfer Theorem:

It states that maximum power will be transferred from a source (represented by its Thevenin's equivalent) to a load, if the load resistance is equal to the source resistance.



Proof: In the figure is shown a source network represented by a voltage source V_s in series with Rs (i.e., by its Thevenin's equivalent circuit).

Across the terminals A, B, a load resistance R_L is connected.

Now current flowing in R_L is $I_L = \frac{V_S}{R_S+R_L}$

power delivered to the load $P_L = I_L^2 \cdot R_L = \frac{V_S^2 \cdot R_L}{(R_S+R_L)^2}$

To find out the value of R_L for which maximum power will be delivered (transferred) to R_L,

we have $\frac{d}{dR_L}(P_L) = 0$.

Now $\frac{d}{dR_L}(P_L) = \frac{d}{dR_L}\left[\frac{V_S^2 \cdot R_L}{(R_S+R_L)^2}\right] = V_S^2\left[\frac{(R_S+R_L)^2 - 2R_L(R_S+R_L)}{(R_S+R_L)^4}\right]$

$= 0$

7

$\Rightarrow (R_S+R_L)^2 - 2R_L(R_S+R_L) = 0$

$\Rightarrow R_S^2 + R_L^2 + 2R_S \cdot R_L - 2R_S \cdot R_L - 2R_L^2 = 0$

$\Rightarrow R_S^2 - R_L^2 = 0$ or, $\boxed{R_S = R_L}$

Hence maximum power will be transfered from source to load, when load resistance is equal to source resistance.

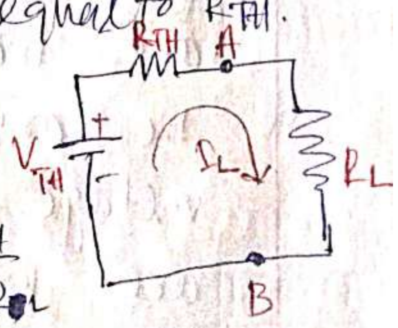
Steps followed while applying Max. power transfer Theorem

Step-I: Remove the load resistance R_L and mark the terminals A, B.

Step-II: Find open circuit voltage across A, B $V_{AB} = V_{TH}$.

Step-III: Find R_{TH} or, R_{AB}, across A, B, looking towards the network by replacing voltage source by short circuit and current source by open circuit.

Step-IV: As per maximum power transfer theorem, for maximum power to be transfered to load, the load resistance R_L will be equal to R_{TH}.



Step-V: Maximum power can be calculated as follow -

current $I_L = \frac{V_{TH}}{R_{TH}+R_L} = \frac{V_{TH}}{2R_{TH}} = \frac{V_{TH}}{2R_L}$

(as $R_{TH} = R_L$)

$P_{max} = I_L^2 \cdot R_L = \frac{V_{TH}^2}{4R_{TH}^2} \cdot R_{TH} = \frac{V_{TH}^2}{4R_{TH}} = \frac{V_{TH}^2}{4R_L}$

(iv) Superposition Theorem: It states that in a linear network containing several sources and resistances, the response (voltage or current) at any point (or in any element), is the

(8) algebraic sum of responses taken separately for each individual source while all other sources are replaced by their respective internal resistances.

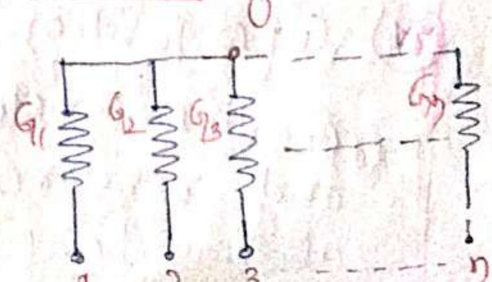
A linear network is one whose parameters are constant i.e, the parameters donot change with change in voltage or current. Or in other words a linear network is one in which the voltage and current have a linear relationship.

An ideal voltage source has zero internal (sc) impedance, so it is replaced by a short circuit, and an ideal current source has infinite internal resistance, so it is replaced by opencircuit (oc)

(V) Millman's Theorem : It states that —

if a network contains a number of admittances $G_1, G_2, G_3 - - G_n$ terminating at a common point O', the other ends of the admittances are marked with 1, 2, 3 - - n and let O be any other point in the network, let the voltage drops from O to 1, O to 2, O to 3, - - - O to n are known (i.e, $V_{01}, V_{02}, V_{03} - - V_{0n}$ are known) then according to Millman's Theorem — voltage drop from O to O' is given by —

$$\boxed{V_{00'} = \frac{V_{01}G_1 + V_{02}G_2 + V_{03}G_3 + - - - V_{0n}G_n}{G_1 + G_2 + G_3 + - - - - + G_n}}$$



proof :

voltage drop across $G_1 = V_{1O'}$

$= V_{00'} - V_{01}$

(9)

current through $G_1 = I_{10'} = V_{10'} \cdot G_1$

$$= (V_{00'} - V_{01}) G_1$$

Similarly $I_{20'} = (V_{00'} - V_{02}) G_2$

$$I_{30'} = (V_{00'} - V_{03}) G_3$$

$$\vdots$$

$$I_{n0'} = (V_{00'} - V_{0n}) G_n$$

By applying KCL to $0'$ —

$$I_{10'} + I_{20'} + I_{30'} + \cdots + I_{n0'} = 0$$

$$\Rightarrow (V_{00'} - V_{01}) G_1 + (V_{00'} - V_{02}) G_2 + \cdots + (V_{00'} - V_{0n}) G_n = 0$$

$$\Rightarrow V_{00'} (G_1 + G_2 + G_3 + \cdots + G_n) = V_{01} G_1 + V_{02} G_2 + \cdots + V_{0n} G_n$$

$$\text{or, } \boxed{V_{00'} = \frac{V_{01} G_1 + V_{02} G_2 + V_{03} G_3 + \cdots + V_{0n} G_n}{G_1 + G_2 + G_3 + \cdots + G_n}}$$

(vi) Reciprocity Theorem: It states that, in any linear network containing several sources and impedances, if a voltage 'V' applied between any two terminals produces a current I in any branch (as measured by a zero impedance ammeter), then the same 'V' and 'I' will be obtained if the position of the voltage source and ammeter are interchanged.

It can also be stated as —

In any linear network containing several sources and impedances, if a current source I applied between any two terminals produces

(10) a voltage V across any branch (as measured by an infinity impedance voltmeter), then the same I and V will be obtained if the positions of the current source and voltmeter are interchanged.

① Unit-3 : **Power Relations in AC Circuits & Transient Response of passive Circuits**

3.1. Definitions:

a) Alternating quantity (current or voltage)

An alternating quantity (current or voltage) is that which varies both in magnitude and polarity (direction) with respect to time.

Most commonly used and naturaly occuring alternating signal is the sinusoidal signal.

A sinusoidal signal (current or voltage) is expressed by the common mathematical notation as—

$$i(t) = I_m \sin(\omega t + \varphi) = I_m \sin(\theta + \varphi) \text{ — for current}$$

$$v(t) = V_m \sin(\theta + \varphi) = V_m \sin(\omega t + \varphi) \text{ — voltage}$$

where $i(t)$ or $v(t)$ → instantaneous values of current or voltage at time 't' respectively.

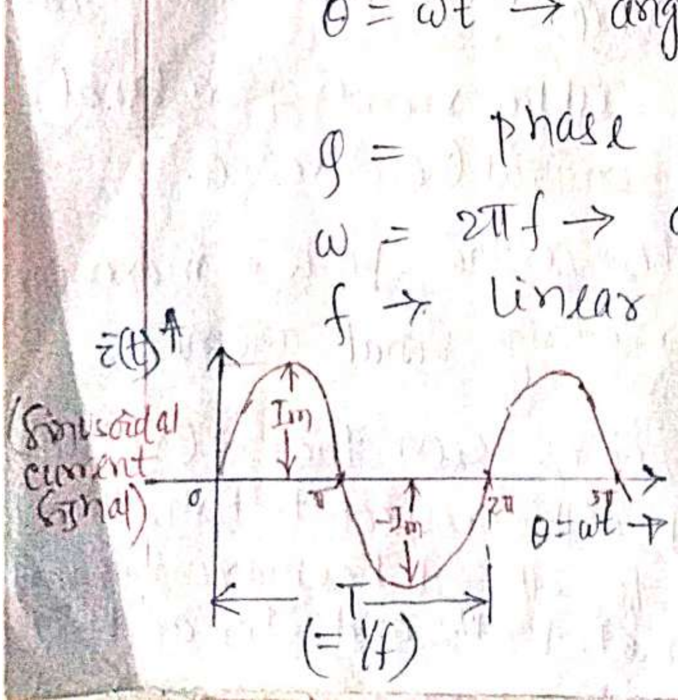
I_m or V_m ⇒ Amplitude, peak or maximum value of current or voltage resp.

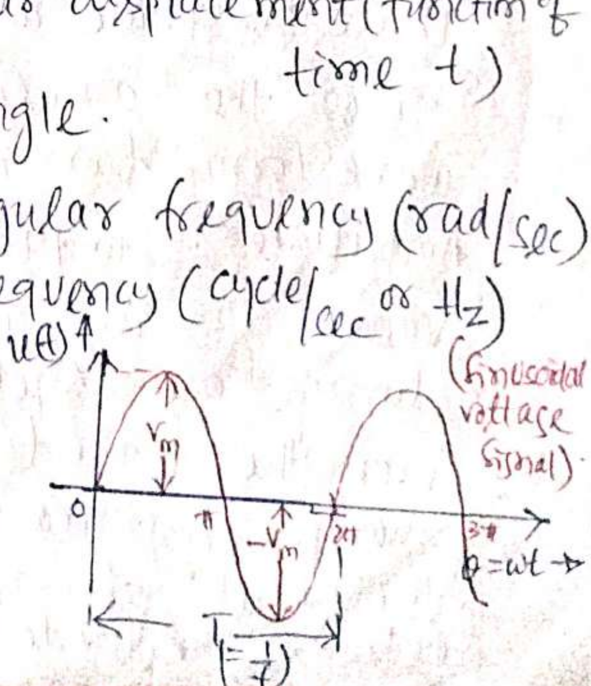
$\theta = \omega t$ → angular displacement (function of time t)

φ = phase angle.

$\omega = 2\pi f$ → angular frequency (rad/sec)

f → linear frequency (cycle/sec or Hz)





(2) Cycle : one complete alternation or repetition of a set of values of an alternating quantity is called one cycle.

Time period (T) : Time required to complete one cycle by the alternating quantity is called periodic time or time period (T).

Frequency (f) : The number of cycles completed in unit time (usually one second) is called the frequency (or linear frequency) of the alternating signal. Unit of frequency is cycle/second or Hertz (Hz).

Angular (or radian) frequency (ω) : From the figure of the sinusoidal signal wave-form, it is seen that the angular displacement $\theta = \omega t$ at the end of one complete cycle is 2π. Since T is the time period i.e., time taken for one cycle to complete then putting $t = T$ in $\omega t = 2\pi$, we have —

$$\omega T = 2\pi \text{ or } \boxed{\omega = \frac{2\pi}{T} = 2\pi f}$$

$\omega \rightarrow$ is the angular frequency. Unit of angular frequency is rad/sec

Phase (or phase angle) (ϕ) :

It is the angular displacement of a signal wave-form from its initial (zero) value.

Amplitude : Amplitude is the peak or maximum value of an alternating signal attained during one cycle.

From the waveform, it is seen that $i(t)$ or $v(t)$ changes continuously with respect to time 't' in the cycle from 0 to 2π. It has maximum value or peak or amplitude I_m at $t = \pi/2$ and $-I_m$ at $t = 3\pi/2$

③ Average (mean) value of an alternating quantity

Average or mean value of an alternating quantity over a given interval of time is the sum of all instantaneous values taken over that interval divided by the time interval.

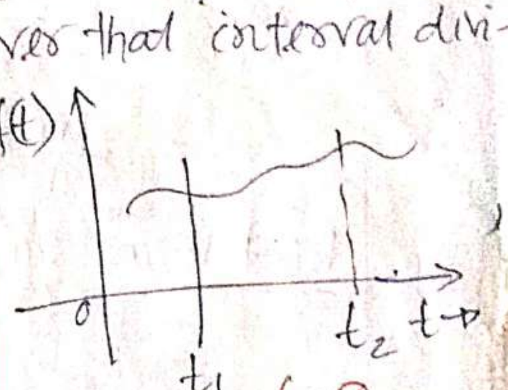


fig ①

- avg. value of the signal $f(t)$ over time interval t_1 to t_2 is $\boxed{f_{av}(t) = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t)\,dt}$

- for a periodic signal $f(t)$ of time period T, the avg value is $\boxed{f_{av}(t) = \frac{1}{T}\int_0^T f(t)\,dt}$

- If the alternating quantity is represented by a curve, then the average value is the ratio of area under the curve to the length of the base of the curve.

$$\boxed{\text{average value} = \frac{\text{area under the curve}}{\text{length of base of curve}}}$$

Average value of a sinusoidal function:

a) complete cycle average:

Let $i(t) = I_m \sin \omega t$

then $I_{av} = \frac{1}{2\pi}\int_0^{2\pi} I_m \sin \omega t \cdot dt$

$= \frac{I_m}{2\pi}\int_0^{2\pi} \sin \omega t \cdot d(\omega t)$

$= \frac{I_m}{2\pi}\left[-\cos \omega t\right]_0^{2\pi} = -\frac{I_m}{2\pi}(1-1) = 0.$

∴ average over one complete cycle of a sinusoidal function is zero.

(4) b) Half cycle average:

Average over a half cycle (0 to π) is given by –

$$I_{av} = \frac{1}{\pi}\int_0^{\pi} i(t)\cdot dt = \frac{1}{\pi}\int_0^{\pi} I_m \sin\omega t \cdot d(\omega t)$$

$$= \frac{I_m}{\pi}\left[-\cos\omega t\right]_0^{\pi} = -\frac{I_m}{\pi}(-1-1) = \frac{2I_m}{\pi} = \frac{I_m}{\pi/2}$$

Root mean square (or effective) value of a sinusoidal signal.

The effective or root mean square value of an alternating current is equal to that value of direct current, which will produce the same amount of heat in the same resistor during same time as produced by the alternating current.

Heat produced ~~in~~ by an alternating current of instantaneous value $i(t)$ in a resistor R during a time interval dt is $i^2(t) \cdot R \cdot dt$. Total heat produced in one cycle (of time period T) is

$$H_{ac} = \int_0^T i^2(t)\cdot R\cdot dt \quad \text{—— (1)}$$

Heat produced by the direct current I_{dc} in the same resistor R, over a time period T is

$$H_{dc} = I_{dc}^2 \cdot R \cdot T \quad \text{—— (2)}$$

Equating (1) & (2) —

$$I_{dc}^2 R\cdot T = \int_0^T i^2(t)\cdot R\cdot dt$$

or $$I_{dc}^2 = \frac{1}{RT}\cdot R\int_0^T i^2(t)\cdot dt$$

or $$\boxed{I_{dc} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\cdot dt} = I_{rms}}$$

$$\therefore \boxed{I_{rms} = I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt}}$$

(5) **rms value of sinusoidal signal (current or voltage)**

Let $i(t) = I_m \sin \omega t$

then $I_{rms}^2 = \frac{1}{2\pi}\int_0^{2\pi} i^2(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} I_m^2 \sin^2 \omega t \, d(\omega t)$

$$= \frac{I_m^2}{2\pi}\int_0^{2\pi}\left(\frac{1-\cos 2\omega t}{2}\right) d\omega t$$

$$= \frac{I_m^2}{4\pi}\left[\omega t - \frac{\sin 2\omega t}{2}\right]_0^{2\pi}$$

$$= \frac{I_m^2}{4\pi}\left[(2\pi - 0) - \frac{1}{2}\underbrace{(\sin 2\omega t)_0^{2\pi}}_{=0}\right]$$

$$= \frac{I_m^2}{4\pi}\left[2\pi\right] = \frac{I_m^2}{2}$$

$$\therefore \boxed{I_{rms} = \frac{I_m}{\sqrt{2}}}$$

Similarly for $v(t) = V_m \sin \omega t$, $\boxed{V_{rms} = \frac{V_m}{\sqrt{2}}}$

Form factor: (K_f) form factor of an alternating signal is defined as the ratio of rms value to average value. $\boxed{K_f = \frac{\text{rms value}}{\text{avg. value}}}$

For a sinusoidal voltage waveforms $v(t) = V_m \sin \omega t$

$V_{rms} = \frac{V_m}{\sqrt{2}}$ and $V_{avg} = \frac{2V_m}{\pi}$

$$\therefore K_f = \frac{V_m/\sqrt{2}}{2V_m/\pi} = \frac{\pi}{2\sqrt{2}} = 1.11$$

Peak factor (K_P): For an alternating signal peak factor is the ratio of peak or maximum value to the rms value.

$$\boxed{K_P = \frac{\text{Peak value}}{\text{rms value}}}$$

for a sinusoidal waveform, $K_P = \frac{I_m}{I_m/\sqrt{2}} = \sqrt{2} = 1.414$.

(6) Instantaneous Power :

The product of instantaneous voltage and instantaneous current is called instantaneous power.

$\boxed{p(t) = v(t) \cdot i(t)}$ where $v(t)$ and $i(t)$ are instantaneous values of voltage & current respectively.

If the network contains only one resistor R,

then $v(t) = R \cdot i(t)$ or $i(t) = \frac{v(t)}{R}$.

then $p(t) = v(t) \cdot i(t) = v(t) \cdot \frac{v(t)}{R} = \frac{v^2(t)}{R}$

$= i(t) \cdot R \cdot i(t) = i^2(t) R$.

Let the applied voltage $v(t) = V_m \sin \omega t$

and current $i(t) = I_m \sin(\omega t + \theta)$

where θ is the angle between voltage and current phasor.

then power $p(t) = v(t) \cdot i(t) = V_m \sin \omega t \cdot I_m \sin(\omega t + \theta)$

applying the formulae
$\sin A \cdot \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$

$= V_m \cdot I_m [\sin(\omega t + \theta) \cdot \sin \omega t]$

$= \frac{V_m \cdot I_m}{2} [\cos(\omega t + \theta - \omega t) - \cos(\omega t + \theta + \omega t)]$

$= \frac{V_m I_m}{2} \cos\theta - \frac{V_m I_m}{2} \cos(2\omega t + \theta)$ —(1)

from eqn (1) it is evident that power $p(t)$ has two terms 1) one is constant $\frac{V_m I_m}{2} \cos\theta$ and 2) 2nd term have an average value of zero.

The first term $\frac{V_m I_m}{2} \cos\theta$ is called real power or average power

$\therefore$ Avg. Power $= \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \cos\theta = V_{rms} I_{rms} \cos\theta$.

where $V_{rms} = \frac{V_m}{\sqrt{2}}$ is the rms value of $v(t)$.

and $I_{rms} = \frac{I_m}{\sqrt{2}}$ is the rms value of $i(t)$

(7) and the term $\cos\theta$ is called power factor.

$\therefore$ Power factor $= \cos\theta$.

where θ is the angle between voltage $v(t)$ and current $i(t)$. In a pure resistive circuit, voltage and current are in phase, so $\theta = 0°$ and $\cos\theta = \cos 0 = 1$.

Apparent (complex) power: Let the rms voltage and current are represented in magnitude and phase angle form as —

$$V_{rms} = |V_{rms}| \angle \theta_v \text{ and } I_{rms} = |I_{rms}| \angle \theta_i$$

then complex power is defined as $S = V_{rms} I_{rms}^*$

$*$ symbol represents complex conjugate. complex conjugate of $a+jb$ is $a-jb$.

So, I_{rms}^* is the complex conjugate of I_{rms}

complex power $S = V_{rms} I_{rms}^*$ is a complex quantity having a real part given by average power.

$P = Re[S] = Re(VI^*) = VI\cos\theta$ is the avg. power.

$Q = Im[S] = Im(VI^*) = VI\sin\theta$ is the reactive power

The complex power $S (= P+jQ)$ is also called apparent power. Average power $\boxed{P = I^2 \cdot R}$.

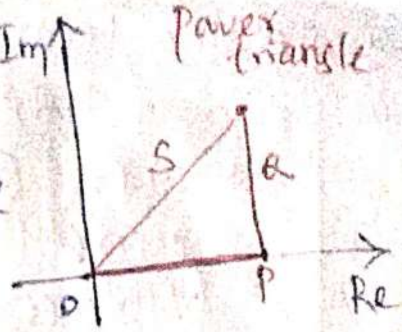
Reactive power $\boxed{Q = I^2 \cdot X}$

Complex power $\boxed{S = I^2 \cdot Z}$

where $R \rightarrow$ resistance, $X \rightarrow$ reactance and $Z \rightarrow$ impedance ($Z = R+jX$).

Power factor $\boxed{pf = \cos\theta = \frac{R}{Z} = \frac{P}{S}}$

(8) Power triangle: Graphical representation of complex power S, as the vector sum of average (active) power P and reactive power Q, is called a power triangle. It is as shown in the figure. If any two quantities out of 3, (S, P, Q) are known, third can be obtained by trigonometric relationships.

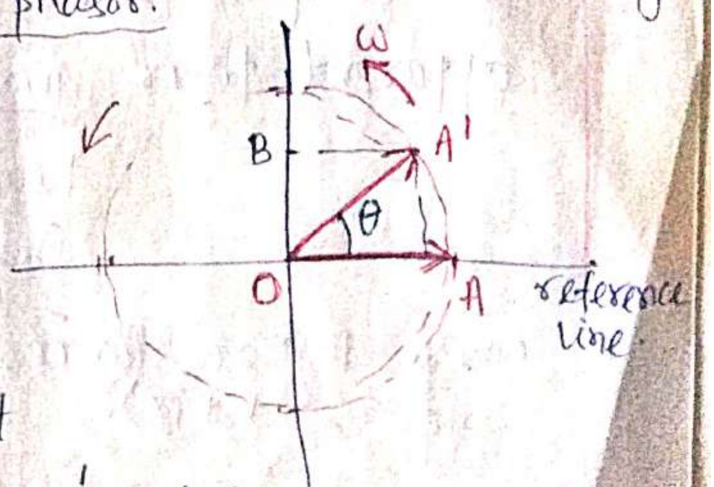


- If the power triangle is in 1st quadrant (as shown in the fig), then the power factor is lagging.
- If the power triangle is in 4th quadrant then power factor is leading.

3.2 phasor Representation of alternating quantity

A sinusoidal quantity may be represented by a line fixed at one end (at centre or origin) and rotating anticlockwise at a constant angular velocity ω radian per second. The length of the line is equal to the peak value of the sinusoid. This rotating line is called the phasor.

Let consider the phasor $OA = I_m$ is rotating with an angular velocity ω rad/sec. After a time t, it



occupies a position OA'. Let OA takes a time t second to rotate through an angle θ, then $\theta = \omega t$

(9) The vertical component of $OA' = OB = OA' \sin \omega t$

$= I_m \sin \omega t = i(t)$

where $i(t)$ is the instantaneous value of the sinusoid. Hence the vertical component of the phasor represents the instantaneous value of the sinusoidal current or voltage.

The phase of an alternating quantity at any time 't' is ~~the~~ defined as the angle which the phasor makes with the reference line.

The general from of equation of a sinusoidal signal is given by $i(t) = I_m \sin(\omega t + \varphi)$

for voltage $v(t) = V_m \sin(\omega t + \varphi)$

where $\varphi \rightarrow$ initial phase angle.

Inphase, leading and lagging ac signals:

Two sinusoidal signals are said to be in phase when they begin and end simultaneously. Also they attain minimum and maximum value at the same time.

When two signals are not in phase, then one is said to be leading (which is ahead in time) and other is lagging (which is delayed in time).

Sinusoidal signal and exponential signals:

Let consider a sinusoidal voltage signal

$$v(t) = V_m \sin(\omega t + \varphi) \text{ —— } (1)$$

By Euler's identity, $e^{j\omega t} = \cos \omega t + j \sin \omega t$

then $V_m e^{j\omega t} = \underbrace{V_m \cos \omega t}_{\text{real part}} + j \underbrace{V_m \sin \omega t}_{\text{imaginary part}}$

$$\therefore v(t) = V_m \sin(\omega t + \varphi) = I_m \{ V_m e^{j(\omega t + \varphi)} \}$$

$$= I_m \{ V_m \cdot e^{j\omega t} \cdot e^{j\varphi} \} \text{ —— } (2)$$

(10) Let $V_m e^{j\varphi} = V_m = |V_m| e^{j\varphi} = |V_m| \angle \varphi$

where $|V_m|$ is the magnitude and φ is the phase angle of the phasor V_m.

Now equation (2) can be written as —

$$v(t) = I_m \{ V_m e^{j\omega t} \} \quad \text{—— (3)}$$

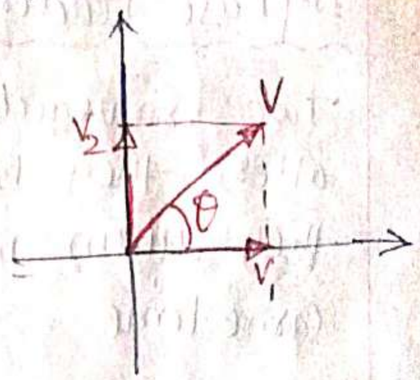
where $e^{j\omega t}$ is a complex exponential for which $|e^{j\omega t}| = \sqrt{\cos^2 \omega t + \sin^2 \omega t} = 1$. So the magnitude of $e^{j\omega t}$ is always 1 (unity)

at $t=0$, $\omega t = 0$, $e^{j\omega t} = e^0 = 1$, at $\omega t = \pi/2$, $e^{j\omega t} = e^{j\pi/2} = 0 + j1$.

at $\omega t = \pi$, $e^{j\omega t} = e^{j\pi} = -1 + j0$, at $\omega t = 2\pi$, $e^{j\omega t} = e^{j2\pi} = 1 = 1 + j0$

Complex number representation of Sinusoidal Signal:

As sinusoids can be represented by phasors and phasors by complex numbers, Hence any phasor V can be represented in 4 different ways as —



a) Rectangular form, $\boxed{V = V_1 + jV_2}$

b) Polar form, $\boxed{V = V \angle \theta}$

c) exponential form, $\boxed{V = V \cdot e^{j\theta}}$

d) Trigonometric form, $\boxed{V = V\cos\theta + jV\sin\theta}$

Phasor Algebra:

a) addition and subtraction of phasors:

Let consider two phasors A and B represented by

$A = a_1 + ja_2$, $B = b_1 + jb_2$

then $\boxed{A+B = (a_1+b_1) + j(a_2+b_2)}$

and $\boxed{A-B = (a_1-b_1) + j(a_2-b_2)}$

(It is easier to perform addition and subtraction in rectangular form)

(ii) b) Multiplication and Division of two phasors.

Let $A = a_1 + ja_2 = A\angle\theta$, $B = b_1 + jb_2 = B\angle\phi$

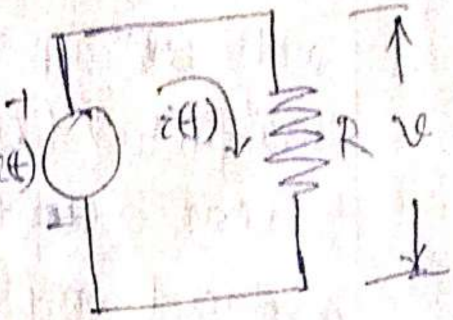
then $\boxed{A \cdot B = A \cdot B \angle\theta+\phi}$

and $\boxed{A/B = \frac{A}{B} \angle\theta-\phi}$

(It is easier to perform multiplication and division in polar form)

3.3. Single phase AC circuits.

a) AC through pure resistors:



A sinusoidal voltage is applied to a pure resistor of R ohms. Let the sinusoidal voltage be $v(t) = V_m \sin\omega t$

$= V_m \sin 2\pi f t = Im\{V_m e^{j\omega t}\}$

Applying KVL to the circuit —

$v = i \cdot R$ or $i = \frac{v}{R} = \frac{V_m}{R}\sin\omega t = I_m \sin\omega t = Img\{I_m e^{j\omega t}\}$

(where $I_m = \frac{V_m}{R}$).

fig ① shows, the waveforms of v and i. where V_m is the peak value of v and $I_m = \frac{V_m}{R}$ is the peak value of i.

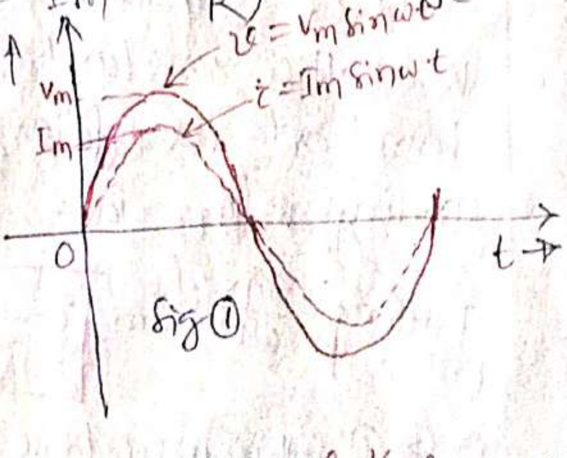


fig ①

fig ② shows the phasors of V and I of voltage and current signals. Both are in phase. Their rms values are $V_{rms} = \frac{V_m}{\sqrt{2}}$ and $I_{rms} = I_m/\sqrt{2}$ respectively.

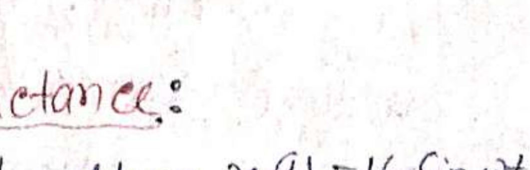


fig ②

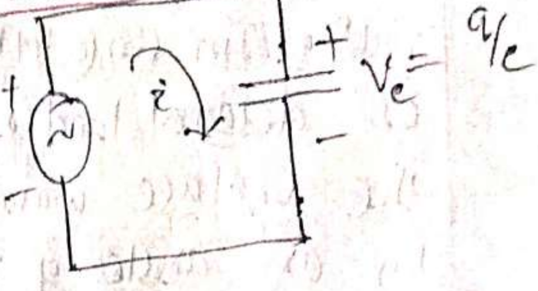
(b) AC through pure inductance:

Let consider a sinusoidal voltage $v(t) = V_m \sin\omega t$ is applied to a circuit consisting a pure

(13) The rms value of voltage is $V_{rms} = \frac{V_m}{\sqrt{2}}$ and rms value of current is $I_{rms} = \frac{V_m}{\omega L\sqrt{2}} = \frac{I_m}{\sqrt{2}}$ where $I_m = \frac{V_m}{\omega L}$. ωL is called the inductive reactance and denoted by X_L (unit is Ω).

$$X_L = \omega L = 2\pi f L$$

(c) AC through pure capacitance



Let consider a sinusoidal voltage $v = V_m \sin \omega t$ is applied to a circuit containing a pure capacitance C farad.

Let the capacitor is charged and at any instant of time 't' charge stored in the capacitor is 'q' coulomb. Then the potential difference between the plates of the capacitor 'v_c' is written as — $v_c = q/c$

Now applying KVL to the circuit —

$v - v_c = 0$ or $v = v_c = q/c$ or $q = c \cdot v$

current in the circuit, $i = dq/dt = \frac{d}{dt} cv$

or, $i = c\frac{d}{dt}(V_m \sin \omega t) = \omega \cdot c \cdot V_m \cos \omega t$

$= \frac{V_m}{1/\omega c} \sin(\omega t + \pi/2) = I_m \sin(\omega t + \pi/2)$

where $\boxed{I_m = \frac{V_m}{1/\omega c}}$ and $\boxed{i = I_m \sin(\omega t + \pi/2)}$

$1/\omega c$ is called capacitive reactance X_c (in Ω)

so, $\boxed{X_c = \frac{1}{\omega c} = \frac{1}{2\pi f c}}$

In the following figures are shown the waveforms of applied voltage and current.

In fig ①, are shown, the waveforms of voltage and current. The voltage waveform of $v(t) = V_m \sin \omega t$ is by solid line and current $i(t) = I_m \sin(\omega t + \pi/2)$ is ~~delayed~~ leading the voltage waveform by an angle of $\pi/2 (=90^\circ)$.

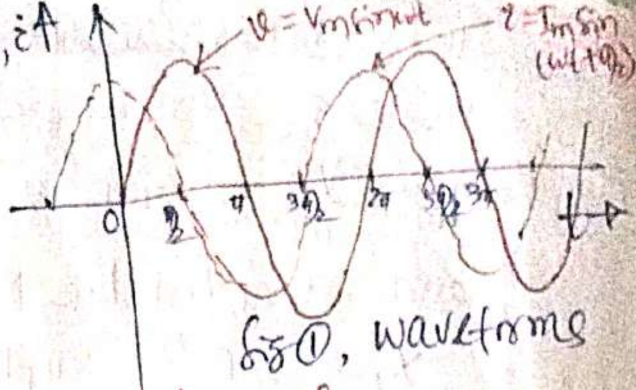


fig ①, waveforms

The same thing is shown in the phasor diagram in which the current phasor leads the voltage phasor by an angle 90°.

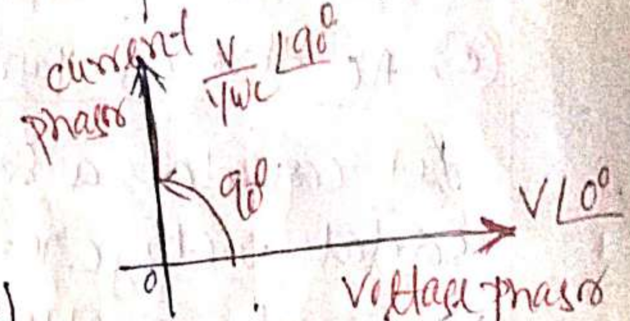
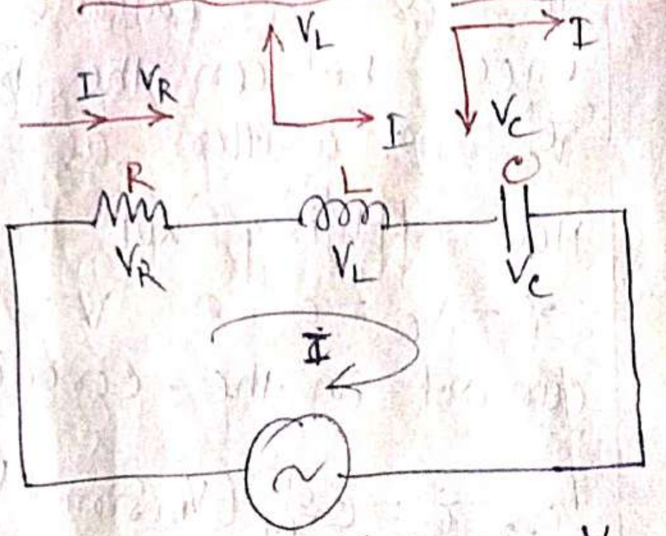


fig ②, phasor diagram

(d) AC analysis of series R-L-C circuit

An ac signal $v = V_m \sin \omega t$ is applied to a series circuit consisting pure resistance R, inductance L and capacitance C.



Current I is taken as reference phasor as it is common to all elements.

Let V_R, V_L and V_c are the voltages across resistor R, inductance L and capacitance C respectively.

(15) Then the applied voltage V is the phasor sum of voltages V_R, V_L and V_C.

$$\therefore \quad V = V_R + V_L + V_C \quad \text{---} \; (1)$$

If Z will be the net impedance of the circuit, then — $V = Z \cdot I$

and $V_R = Z_R \cdot I = R \cdot I$, $V_L = Z_L \cdot I = jX_L \cdot I$

and $V_C = Z_C \cdot I = -jX_C \cdot I$.

putting in eqn (1), $V = RI + jX_L I - jX_C I$

$$= I[R + jX_L - jX_C]$$

$$= I[R + j(X_L - X_C)]$$

$$= I \cdot \{R + jX\}$$

where $X = X_L - X_C$ is the net reactance

$$\therefore \quad \boxed{Z = \frac{V}{I} = R + jX = R + j(X_L - X_C) = Z\angle\varphi}$$

where $\boxed{Z = \sqrt{R^2 + X^2} = \sqrt{R^2 + (X_L - X_C)^2}}$

and $\boxed{\varphi = \tan^{-1}\left(\frac{X}{R}\right) = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)}$

Phasor diagram of series R-L-C circuit:

In a series R-L-C circuit, current I is common to all elements (R, L & C) and therefore is taken to be the reference phasor. The voltages, V_R is inphase with I, V_L is leading I by an angle of 90^0, V_C is lagging behind I by an angle of 90^0.

Case-I: when $V_L > V_C$ i.e, $X_L > X_C$

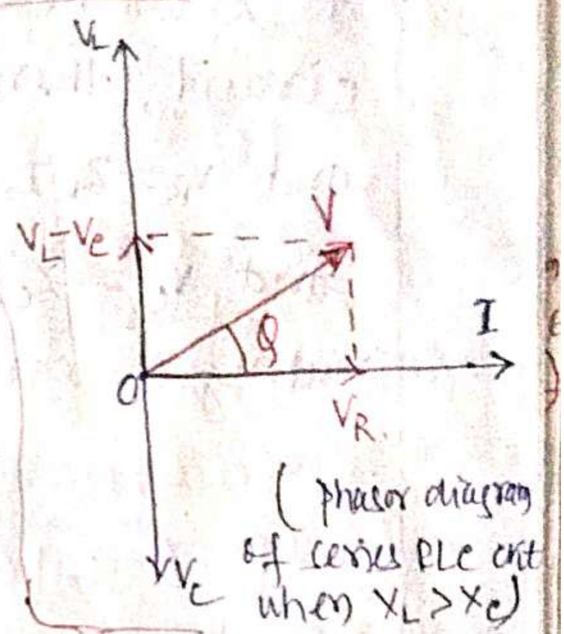
As inductive reactance (X_L) is more than capacitive

(16) reactance(X_C), the circuit is called inductive. V_L and V_C being in opposite direction, the resultant phasor $(V_L - V_C)$ is found and the resultant voltage phasor V is the phasor sum of V_R and $(V_L - V_C)$

$$\therefore V^2 = V_R^2 + (V_L - V_C)^2$$

$$= (I \cdot R)^2 + \{I(X_L - X_C)\}^2$$

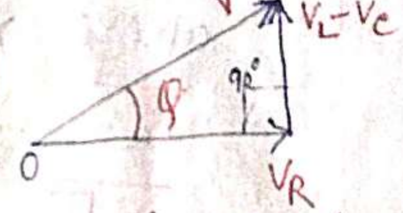
$$\therefore \boxed{V = I\sqrt{R^2 + (X_L - X_C)^2}}$$



(phasor diagram of series RLC cKt when $X_L > X_C$)

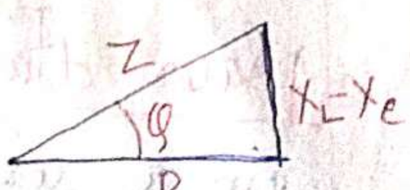
Voltage triangle:

Voltage triangle is as shown in figure, in which the resultant phasor V is the phasor sum of resistance phasor voltage V_R and net reactance voltage phasor $(V_L - V_C)$



(voltage triangle)

Impedance triangle:



(impedance triangle)

In impedance triangle, the resistance R and net reactance $X_L - X_C$ are taken as the two sides of a right angled triangle where as the net impedance Z $[= R + j(X_L - X_C)]$ is taken along the hypotenues.

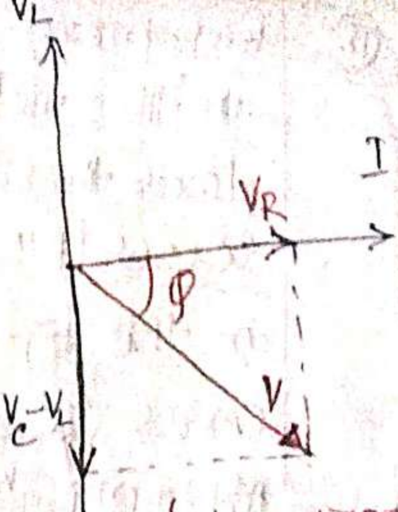
(impedances are not vectors, so no arrow mark is given in impedance triangle)

(17)

case II : when $X_L < X_C$:

As capacitive reactance is more than inductive reactance, the circuit is called capacitive. The resultant voltage V is the phasor sum of V_R and $(V_C - V_L)$, the resultant phasor V, lags behind the current I, by an angle φ given by

$$\boxed{\varphi = \tan^{-1}\left(\frac{X_C - X_L}{R}\right)}$$

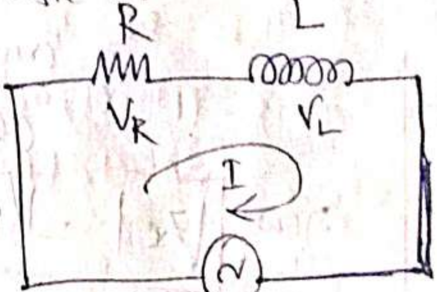


(Phasor diagram of series RLC ckt when $X_C > X_L$)

Voltage triangle & impedance triangle ?

Series R-L circuit :

In the figure is shown a series RL circuit.



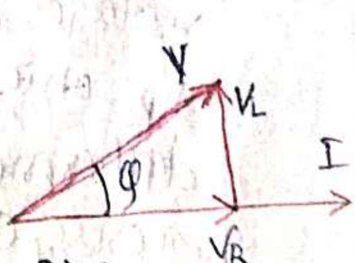
$v = V_m \sin\omega t = V$

Applying KVL to the circuit,

$$V = V_R + V_L = IR + I \cdot j \cdot X_L$$
$$= I[R + jX_L] = I \cdot Z_L$$

where $Z_L = R + jX_L$ = impedance of the circuit

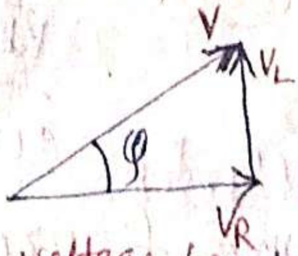
and $X_L = \omega \cdot L = 2\pi f \cdot L$ is the inductive reactance of the ckt.



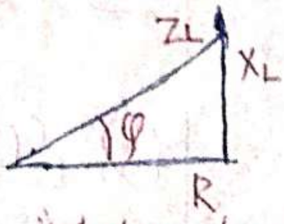
phasor diagram.

Now $\boxed{|Z_L| = \sqrt{R^2 + X_L^2}}$ and $\boxed{\varphi = \tan^{-1}\left(\frac{X_L}{R}\right)}$

In the voltage triangle, the applied voltage V is the phasor sum of V_R and V_L and in impedance triangle resistance R and inductive reactance X_L are the two sides and impedance Z_L is on the



voltage triangle

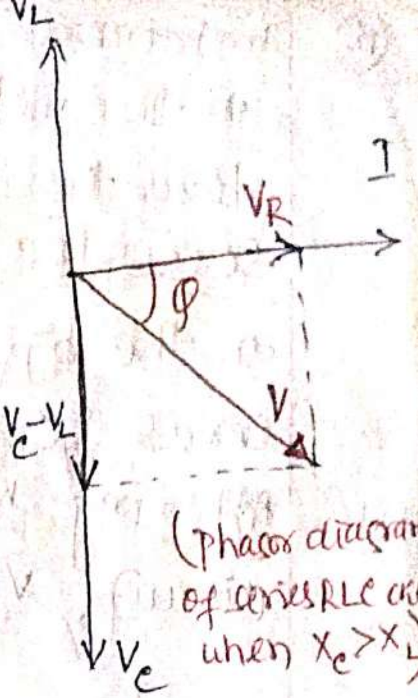


impedance triangle

(17)

case-II: when $X_L < X_C$:

As capacitive reactance is more than inductive reactance, the circuit is called capacitive. The resultant voltage V is the phasor sum of V_R and $(V_C - V_L)$, the resultant phasor V, lags behind the current I, by an angle φ given by $\boxed{\varphi = \tan^{-1}\left(\frac{X_C - X_L}{R}\right)}$

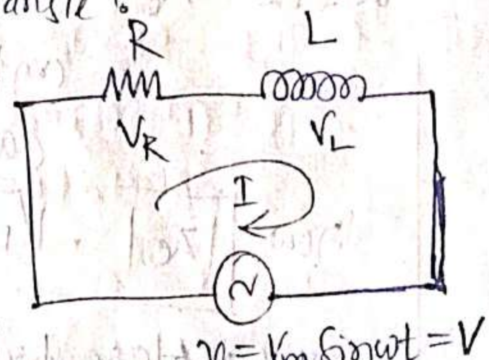


(phasor diagram of series RLC ckt when $X_C > X_L$)

Voltage triangle & impedance triangle?

Series R-L circuit:

In the figure is shown a series RL circuit.

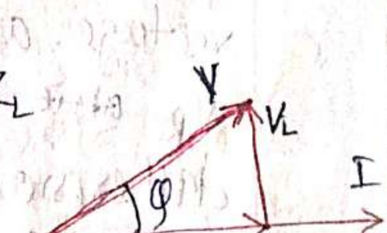


Applying KVL to the circuit,

$$V = V_R + V_L = IR + I \cdot jX_L$$
$$= I[R + jX_L] = I \cdot Z_L$$

where $Z_L = R + jX_L$ = impedance of the circuit

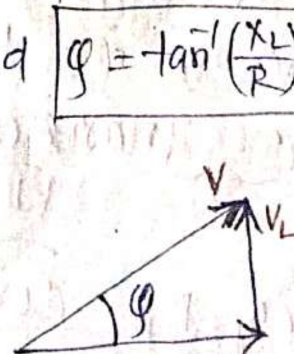
and $X_L = \omega \cdot L = 2\pi f \cdot L$ is the inductive reactance of the ckt.



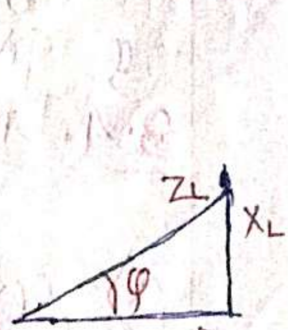
Phasor diagram.

Now $\boxed{|Z_L| = \sqrt{R^2 + X_L^2}}$ and $\boxed{\varphi = \tan^{-1}\left(\frac{X_L}{R}\right)}$

In the voltage triangle, the applied voltage V is the phasor sum of V_R and V_L and in impedance triangle resistance R and



voltage triangle



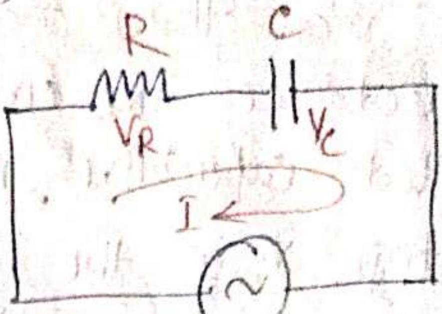
impedance triangle

inductive reactance X_L are the two sides and impedance Z_L is in the

(18) hypotenuse.

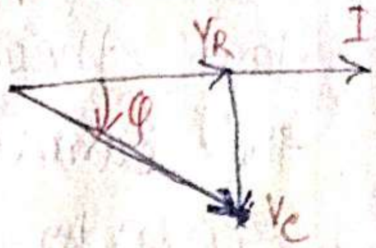
In the voltage triangle, the resultant voltage V leads the current I by an angle $\varphi = \tan^{-1}\left(\frac{X_L}{R}\right)$.

Series RC circuit:



In the fig is shown a series RC circuit. Applying KVL to the circuit,

$$V = V_R + V_C$$
$$= I \cdot R + I \cdot (-jX_C)$$
$$= I[R - jX_C]$$

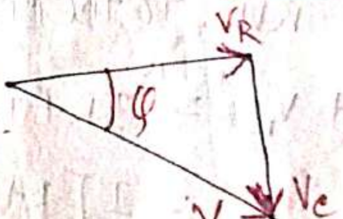


(phasor diagram)

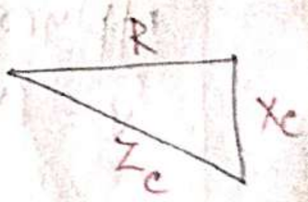
where $Z_C = R - jX_C$ is the net impedance of the circuit, which is capacitive in nature.

now $\boxed{|Z_C| = \sqrt{R^2 + X_C^2}}$ and $\boxed{\varphi = \tan^{-1}\left(\frac{-X_C}{R}\right)}$

In the voltage triangle voltage across resistor V_R ~~that~~ is in 90° phase difference with voltage across capacitor V_C.



voltage triangle

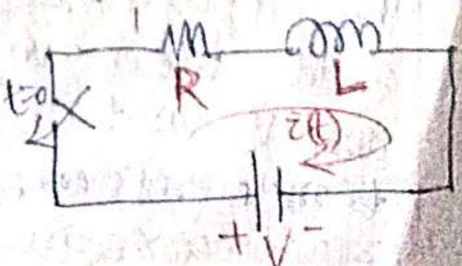


impedance triangle

and the resultant ~~vect~~ voltage V lags V_R (or the current) by an angle φ given by $\varphi = \tan^{-1}\left(\frac{-X_C}{R}\right)$.

3.4. DC - Transient Behaviour of R-L, RC and RLC series ckts with time constant

a) **Series RL circuit:**



In the figure is shown a series RL

(19) circuit connected to a dc voltage source of V volts. The switch is closed at time $t=0$. the inductor was initially unenergised.

For time $t>0$, applying KVL to the circuit –

$$V = Ri + L\cdot\frac{di}{dt} \quad \text{——} \quad (1)$$

eqn (1) is a 1st order linear differential equation and can be solved for current i (by variable separable method).

Now $V = Ri + L\cdot\frac{di}{dt}$ or, $\frac{di}{dt} = \frac{V - R\cdot i}{L}$

or, $\frac{L}{V-R\cdot i}\cdot di = dt$

integrating both sides

$$L\int\frac{di}{V-Ri} = \int dt$$

Put $V-Ri = x$
or, $d(V-Ri) = dx$
or, $-dRi = dx$
or $-R\cdot di = dx$

or $-\frac{L}{R}\int\frac{-R\cdot di}{V-Ri} = \int dt$

or, $-\frac{L}{R}\int\frac{dx}{x} = \int dt$ $\quad \because \int\frac{dx}{x} = \log_e x$

or, $-\frac{L}{R}\log_e(V-Ri) = t + K$ —— (2), where K is an arbitrary constant whose value can be found out from initial condition

At time $t=0$, current in the circuit is zero as the inductor does not allow sudden change in current through it.

Putting $i=0$ at time $t=0$ in eqn (2)

$$-\frac{L}{R}\log_e V = K \quad \text{——} \quad (3)$$

putting this value of K in eqn (2) –

$$-\frac{L}{R}\log_e(V-Ri) = -\frac{L}{R}\log_e V + t$$

(20) or, $-\frac{L}{R}\left[\log_e(V-Ri) - \log_e V\right] = t$

or, $-\frac{L}{R}\log_e\left(\frac{V-Ri}{V}\right) = t$

or, $\log_e\left(\frac{V-Ri}{V}\right) = -\frac{R}{L}\cdot t$

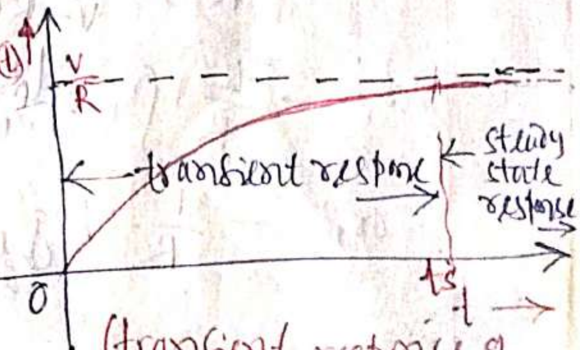
or, $\frac{V-Ri}{V} = e^{-\frac{R}{L}\cdot t}$ or, $V-R\cdot i = V\cdot e^{-\frac{R}{L}\cdot t}$

or, $R\cdot i = V - V\cdot e^{-\frac{R}{L}t}$

or, $\boxed{i = \frac{V}{R} - \frac{V}{R}e^{-\frac{R}{L}\cdot t}}$ —(1) or, $\boxed{i = \frac{V}{R}\left(1-e^{-\frac{R}{L}\cdot t}\right)}$ —(1)

eqn (1) states that, the complete solution or response (i.e., current) of the series R-L circuit consists of two parts (i) steady state response or forced response or zero state response $\left(\frac{V}{R}\right)$ and (ii) transient response or natural response or zero input response $\left(-\frac{V}{R}e^{-\frac{R}{L}\cdot t}\right)$

In the figure is shown the response curve i.e., $i(t)$ versus time t graph.



(transient response of series R-L circuit)

At time $t=0$, current $i(t)=0$, and as time goes on, the response increases and after some time (called settling time) it reaches a steady state value. This time period is called transient period (the response is transient response) after that steady state period and the response is steady state response.

Time constant:

$i(t) = \frac{V}{R}\left[1 - e^{-\frac{t}{L/R}}\right]$ is the response equation or, equation for current.

The term (L/R) in the current equation is called time constant i.e, $\boxed{T = L/R}$

The response after one time constant T, can be found by putting $t = T$ in eqn (4);—

$$i = \frac{V}{R}\left[1 - e^{-t/(L/R)}\right] = \frac{V}{R}\left[1 - e^{-t/T}\right]$$

putting $t = T$, $i = \frac{V}{R}\left[1 - e^{-T/T}\right] = \frac{V}{R}\left[1 - e^{-1}\right]$

$$= \frac{V}{R}\left[1 - \frac{1}{e}\right] = 0.632\frac{V}{R}$$

or, $i = 0.632\frac{V}{R}$ or 63.2% of final steady value

Hence time constant can be defined as the time after which the response reaches 63.2% of its final steady value, starting from zero initial value. i.e, at $t = T$, $i(T) = \left(1 - e^{-T/T}\right) = \frac{V}{R}\left(1 - \frac{1}{e}\right)$

$$= 0.632\frac{V}{R}$$

Similarly value of $i(t)$ at 2 time constant ($2T$), $3T$, $4T$ or $5T$ etc can be found out. for example at $t = 5T$ (5 time constant) —

$$i(5T) = \frac{V}{R}\left(1 - e^{-\frac{5T}{T}}\right) = \frac{V}{R}\left(1 - e^{-5}\right) = 0.993\frac{V}{R}$$

Now voltages across resistor (V_R) and inductor (V_L) can be found out from eqn (4) representing $i(t)$.

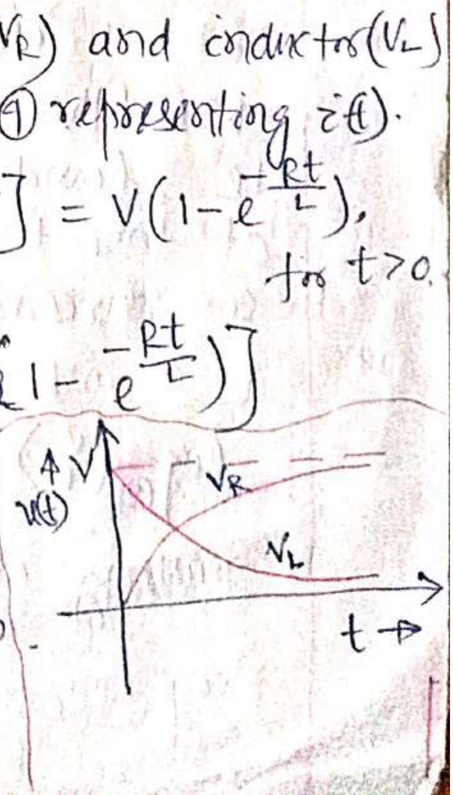
$$V_R = i(t)\cdot R = R\cdot\frac{V}{R}\left[1 - e^{-\frac{R}{L}\cdot t}\right] = V\left(1 - e^{-\frac{Rt}{L}}\right), \text{ for } t > 0.$$

or, $\boxed{V_R = V\left(1 - e^{-Rt/L}\right)}$, for $t > 0$

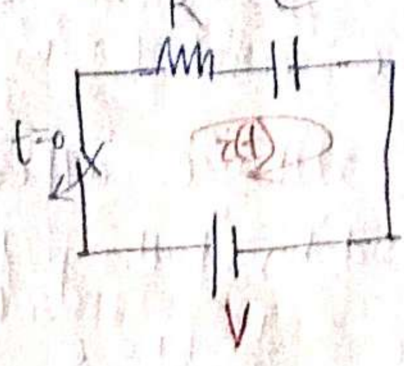
and $V_L = L\cdot\frac{di}{dt} = L\cdot\frac{d}{dt}\left[\frac{V}{R}\left(1 - e^{-\frac{Rt}{L}}\right)\right]$

$$= -\frac{L\cdot V}{R}\left(-R/L\right) e^{-\frac{R}{L}\cdot t}$$

or $\boxed{V_L = V\cdot e^{-\frac{R}{L}\cdot t}}$ for $t > 0$.



(22) (b) Series R-C circuit :



In the figure is shown a series RC circuit connected to a dc voltage V. The switch is closed at time $t=0$.

The capacitor was initially uncharged.

Applying KVL to the circuit for $t>0$

$$V = Ri + \frac{1}{C}\int_0^t i \cdot dt$$

current voltage relation in a capacitor is $i = C\cdot\frac{dv}{dt}$ or $v = \frac{1}{C}\int i dt$

differentiating both sides we get $0 = R\cdot\frac{di}{dt} + \frac{1}{C}\cdot i$

$$\text{or, } \frac{di}{dt} + \frac{1}{CR}\cdot i = 0 \quad \text{—— (1)}$$

This is a first order linear differential equation and can be solved by separating the variables —

from eqn (1) $\frac{di}{i} = -\frac{1}{RC}\cdot dt$

Integrating both sides, $\int\frac{di}{i} = -\frac{1}{RC}\int dt$

$$\text{or, } \log_e i = -\frac{1}{RC}t + K \quad \text{—— (2)}$$

K is the integration constant whose value can be found from initial conditions.

at time $t=0$, capacitor acts as short circuit so current in the circuit $i(0) = \frac{V}{R}$. putting $t=0$ and $i(0) = \frac{V}{R}$ in eqn (2)

$\log_e\left(\frac{V}{R}\right) = 0 + K$ or $K = \log_e\left(\frac{V}{R}\right)$

putting in eqn (2), $\log_e i = -\frac{1}{RC}t + \log_e\frac{V}{R}$

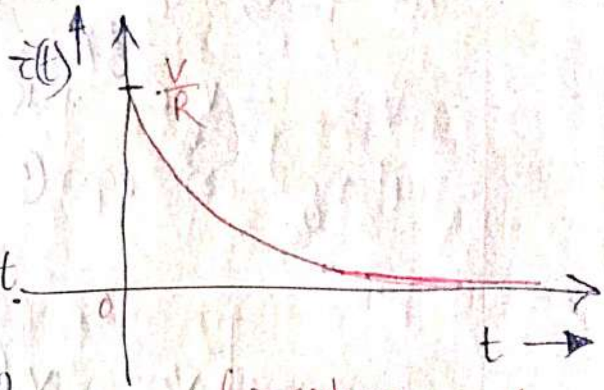
or, $\log_e i - \log_e\frac{V}{R} = -\frac{1}{RC}\cdot t$

or, $\log_e\left(\frac{i}{V/R}\right) = -\frac{1}{RC}\cdot t$

or, $\frac{i}{V/R} = e^{-\frac{1}{RC}t}$ or $\boxed{i = \frac{V}{R} e^{-\frac{t}{RC}}}$ — (4)

or, $\boxed{i = \frac{V}{R} e^{-t/T}}$ where $T = RC$ called time constant

In the figure is shown the response of series RC circuit i.e, graph of $i(t)$ versus time t.



(response of series R-C circuit)

The response decreases from its initial maximum value $\left(\frac{V}{R}\right)$ at time $t=0$ to final zero value at time $t=\infty$.

Time constant:

The term RC in the response equation $i(t) = \frac{V}{R} e^{-t/RC}$ is called time constant T

$\therefore \boxed{T = RC}$

at time $t = T$ (one time constant), the response can be found out as – $i(T) = \frac{V}{R} e^{-T/T} = \frac{V}{R}(e^{-1})$

$= \frac{V}{R}\left(\frac{1}{e}\right) = 0.368\, V/R$

$= 36.8\%$ of initial maximum value.

Hence time constant can be defined as the time taken by the response to decrease to 0.368 or 36.8 % of its initial maximum value at $t=0$.

Now voltage across resistor R (i.e, V_R) and capacitor C (i.e, V_C) can be found from eqn (4)

$V_R = i \cdot R = R \cdot \frac{V}{R} e^{-t/T} = V \cdot e^{-t/T}$ $\boxed{V_R = V \cdot e^{-t/RC}}$

Similarly voltage across the capacitor V_c is

$$V_c = \frac{1}{c}\int i\,dt = \frac{1}{c}\int \frac{V}{R} e^{-t/T} dt$$

$$= \frac{V}{T}\left(-\frac{1}{1/T}\right) e^{-t/T} = -V e^{-t/T} + K \quad \text{--- (5)}$$

value of K can be found out from initial conditions

at $t=0$, $V_c = V$, putting in (5)

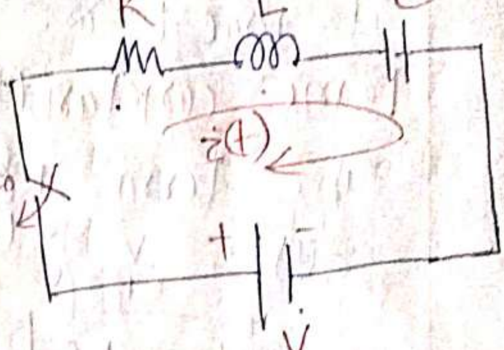
$$V = -V e^{-0/T} + K = -0 + K \quad \text{or } K = V$$

putting in eqn (5), $V_c = -V e^{-t/T} + V$

$$= V\left[1 - e^{-t/T}\right]$$

$$\therefore \boxed{V_c = V\left(1 - e^{-t/T}\right) = V\left(1 - e^{-t/RC}\right)}$$

(c) Transient response of series R-L-C circuit



In the figure is shown a series R-L-C circuit connected to a dc voltage of V volts. switch is closed at $t=0$. The inductor and the capacitor were initialy uncharged. Now applying KVL to the circuit at $t>0$,

$$V = Ri + L\frac{di}{dt} + \frac{1}{c}\int i\,dt$$

differentiating $\quad 0 = R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{1}{C} i$

$$\text{or,}\quad \frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{CL} i = 0 \quad \text{--- (1)}$$

This is a 2nd order differential equation. The auxiliary or characteristic equation for this eqn can be given as $s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC} = 0$ --- (2)

The two roots of eqn (2) are, $s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$

Note: for a quadratic eqn $ax^2 + bx + c = 0$, the roots x_1, x_2 are given by $x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Let $\alpha = \frac{R}{2L}$ and $\beta = \sqrt{\alpha^2 - \omega_0^2}$ where $\omega_0 = \frac{1}{\sqrt{LC}}$

so that $s_1, s_2 = -\alpha \pm \beta$ i.e, $s_1 = -\alpha + \beta$, $s_2 = -\alpha - \beta$ — (3)

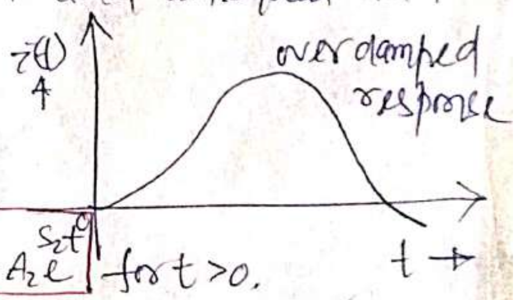
Now the solution of the 2nd order differential eqn (1) can be given as $\boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$ — (4)

where A_1, A_2 are constants whose values can be determined from the values of s_1 and s_2 which are the roots of the equation given in (3)

Now depending upon the values of α and ω_0 (i.e, β), 3 cases for response $i(t)$ are possible

<u>Case-I</u> : when $\alpha > \omega_0$ i.e, $\frac{R}{2L} > \frac{1}{\sqrt{LC}}$

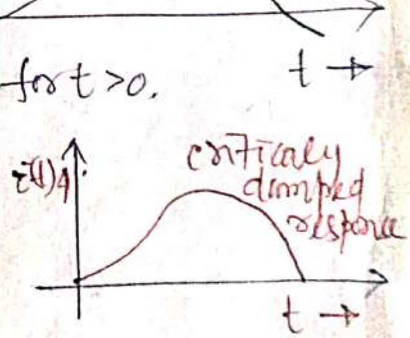
The roots s_1 and s_2 are real and unequal and it gives an overdamped response and the solution is given by —



$$\boxed{i(t) = e^{-\alpha t}\left(A_1 e^{\beta t} + A_2 e^{-\beta t}\right) = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \text{ for } t > 0.$$

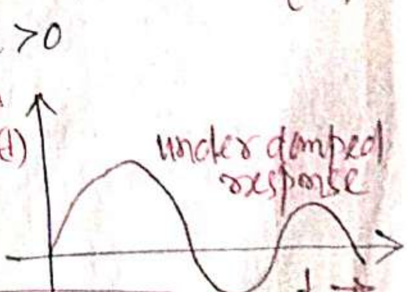
<u>Case-II</u> : when $\alpha = \omega_0$ i.e, $\frac{R}{2L} = \frac{1}{\sqrt{LC}}$.

The roots are real and equal and it gives a <u>critically damped response</u> given by $\boxed{i(t) = e^{-\alpha t}(A_1 + A_2 t)}$ for $t > 0$



<u>Case-III</u> : when $\alpha < \omega_0$ i.e, $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$

The roots are complex conjugate and it gives an <u>under damped response</u> —



$\boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$ where $\boxed{s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}}$

Let $\alpha^2 - \omega_0^2 = \sqrt{-1}\sqrt{\omega_0^2 - \alpha^2} = j\omega_d$ where $j = \sqrt{-1}$, $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

so that $i(t) = e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right) = e^{-\alpha t}\left\{(A_1 + A_2)\left(\frac{e^{j\omega_d t} + e^{-j\omega_d t}}{2}\right)\right.$

$\left. + j(A_1 - A_2)\left(\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{2}\right)\right\} = e^{-\alpha t}\left\{(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t\right\}$

or, $\boxed{i(t) = e^{-\alpha t}\left(B_1 \cos\omega_d t + B_2 \sin\omega_d t\right)}$

where $B_1 = A_1 + A_2$ and $B_2 = j(A_1 - A_2)$

Ch:4 # Resonance ①

① **Resonance:** Resonance is defined as the condition in a circuit containing at least one inductor and one capacitor when the supply voltage and produced current are in phase. So at resonance the equivalent impedance of the circuit is purely resistive. So, the net impedance or admittance of the cut. are real quantities.

For ex: Let the net impedance and/or admittance of a cut at resonance (I) are $Z = R + jX$ or $Y = G + jB$

then at resonance a) $Im(Z) = X = 0$ and $Im(Y) = B = 0$.

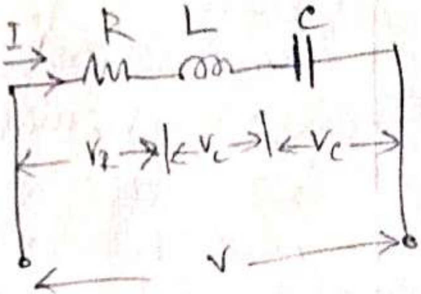
(II) if $Z = Z\angle\varphi$ and $Y = Y\angle\theta$.

then at resonance (b) $\varphi = 0$ and $\theta = 0$.

(III) if $Z = \frac{a+jb}{c+jd}$ then at resonance (c) $\frac{b}{a} = \frac{d}{c}$.

Similarly if $Y = \frac{l+jm}{p+jq}$ then at resonance $\frac{m}{l} = \frac{q}{p}$.

② **Resonance in series RLC cut**



In the figure is shown a series RLC cut, in which R in (Ω), inductor with inductance L (Henry) and capacitance C (Farad) are connected across a voltage V (volts).

inductive reactance $X_L = \omega L = 2\pi f L$

capacitive reactance $X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$.

net reactance $= X = X_L - X_C$.

Total impedance $Z = \sqrt{R^2 + (X_L - X_C)^2}$

A) **condition of resonance**: Let the resonant frequency be f_0 so that $\omega_0 = 2\pi f_0$ and reactances at resonance are $X_{L_0} = 2\pi f_0 L$ and $X_{C_0} = \frac{1}{2\pi f_0 C}$.

(2) The conditions for resonance a) apply voltage and current are in phase b) impedance of the circuit is purely resistive c) net reactance is zero.

so, at resonance $X_{L0} - X_{C0} = 0$, or $\boxed{X_L = X_C}$

and $\boxed{Z_0 = R}$.

at resonance the ckt impedance is equal to the circuit resistance and is called dynamic impedance (Z_0).

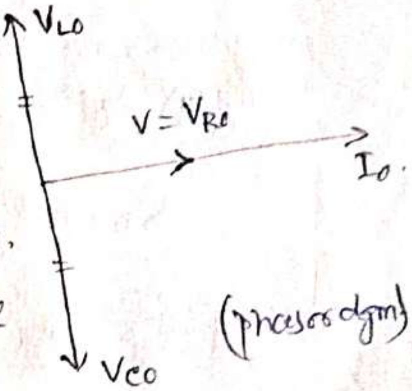
If I_0 is the current at resonance, then $V_R = I_0 \cdot R$ and $V_L = I_0 X_{L0}$ and $V_C = I_0 \cdot X_{C0}$.

So, net voltage $V = \sqrt{V_R^2 + (V_L - V_C)^2} = \sqrt{V_R^2} = V_R = R \cdot I_0$.

If φ_0 is the phase difference between voltage and current at resonance, then $\varphi_0 = 0$ and power factor $\boxed{\cos\varphi_0 = \cos 0 = 1}$ which is maximum at resonance.

The power at resonance $P_0 = VI_0 \cos\varphi_0 = V \times \frac{V}{R} \times 1 = \frac{V^2}{R}$.

or $\boxed{P_0 = \frac{V^2}{R}}$



(phasor dgm)

(ii) Phasor diagram at resonance :

voltage across inductor $(V_{L0}) = X_{L0} I_0$ leads the current I_0 by 90^0 and $V_{C0} = X_{C0} I_0$ lags behind the current I_0 by 90^0. Both are (V_{L0} & V_{C0}) equal in magnitude and opposite in direction. So net voltage $V = V_{R0}$, which is in phase with current.

(iii) Resonant frequency : (f_0) at resonant frequency (f_0), net reactance is zero, i.e, $X_{L0} = X_{C0}$ or $2\pi f_0 L = \frac{1}{2\pi f_0 C}$

or $f_0^2 = \frac{1}{(2\pi)^2 LC}$ or $\boxed{f_0 = \frac{1}{2\pi\sqrt{LC}}}$ Hz.

and $\boxed{\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{LC}}}$ rad/s

Variation of different quantities with frequency

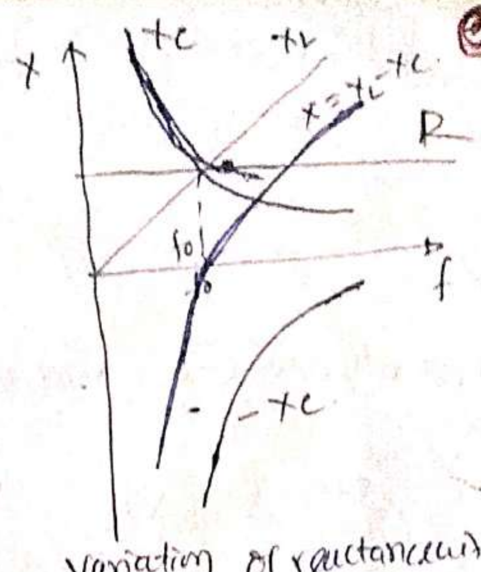
a) inductive reactance $X_L = 2\pi f L$ is directly proportional to f. Hence it is a st. line passing through the origin.

b) capacitive reactance $X_C = \frac{1}{2\pi f C}$ is inversely proportional to f and hence it is a rectangular hyperbola. graph of X_L vs f and X_C vs f cut at a frequency $f = f_0$ (at resonance).

c) variation of net reactance $X = X_L - X_C$. It is obtained from the graph of X_L vs f and $-X_C$ versus f, which is a hyperbola.

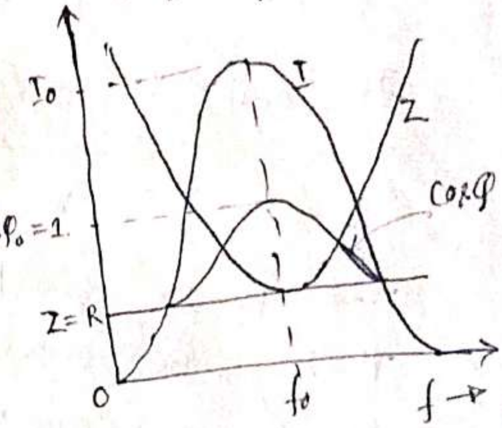
d) variation of resistance (R): It is independent of frequency f and hence a horizontal st. line.

e) variation of impedance (Z). It is minimum at resonance given by $Z = R$ and increases on either side of resonant frequency. At frequencies $f < f_0$, Z is capacitive and $(X_C > X_L)$, so power factor is lagging and for $f > f_0$, Z is inductive $(X_L > X_C)$, so power factor is leading and at resonant frequency $\cos\phi_0 = 1$ (maximum).



Variation of reactance with frequency.

f) Current: current has a maximum value at resonance $I_0 = V/R$. and its value decreases on either side of resonant freq. The current versus freq. curve is called resonance curve or response curve.



③ Properties of a second order series RLC resonant ckt:

A ckt consisting of series connections of R (in Ω), inductor L (in Henry) and capacitance C (in Farad) is called a second order series resonant ckt. The properties of such ckts are -

a) resonant freq. $\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$ (rad/s) and $\boxed{f_0 = \frac{1}{2\pi\sqrt{LC}}}$ (Hz).

b) at resonance ($f = f_0$), the ckt. acts like a resistor, above resonance ($f > f_0$), the ckt acts like RL and below resonance ($f < f_0$) the ckt acts like RC.

c) if resistance is '0', then the ckt acts like short ckt.

d) bandwidth $\boxed{B.W. = R/L}$

e) Quality factor $\boxed{Q_{0s} = \frac{1}{R}\sqrt{L/C}}$

④ Quality factor (Q_0):

Quality factor Q is defined as $Q = 2\pi \times \frac{\text{max. stored energy per cycle}}{\text{energy dissipated per cycle}}$

(4) now total energy stored in a series RLC ckt is - $W_s = \frac{1}{2}Li^2 + \frac{1}{2}C v_c^2$ — (1)

If $i = I_m \sin\omega t$, voltage across the capacitor $v_c = \frac{1}{C}\int i\,dt$ (2)

$$= -\frac{I_m}{\omega C}\cos\omega t \quad \text{— (2)}$$

$$\therefore W_s = \frac{1}{2}L \cdot I_m^2 \sin^2\omega t + \frac{1}{2}C \cdot \frac{I_m^2}{\omega^2 C^2}\cos^2\omega t \quad \text{— (3)}$$

at resonance let max value of current by I_{mo}, ~~at ω_0~~ and $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$

$$\text{So, } W_s = \frac{1}{2}L I_{m_0}^2 \cdot \sin^2\omega_0 t + \frac{1}{2}C \cdot I_{m_0}^2 \times \frac{1}{\frac{1}{LC}\times C^2}\cdot\cos^2\omega_0 t$$

$$= \frac{1}{2}L \cdot I_{m_0}^2\left(\sin^2\omega_0 t + \cos^2\omega_0 t\right) = \frac{1}{2}\cdot L \cdot I_{m_0}^2$$

If I_0 is the rms current at resonance then $I_0 = \frac{I_{mo}}{\sqrt{2}}$ and $I_0^2 = \frac{I_{mo}^2}{2}$.

$$\therefore \boxed{W_s = L I_0^2} \quad \text{— (4)}$$

now power dissipated in the ckt, at resonance, $P_{do} = I_0^2 \cdot R$.

This is the energy dissipated per second, so energy dissipated per cycle at resonance is $W_{do} = \frac{P_{do}}{f_0} = \frac{I_0^2 \cdot R}{\omega_0/2\pi} = \frac{2\pi I_0^2 \cdot R}{\omega_0}$ — (5).

Hence Q factor of the series RLC resonant ckt, $Q_{os} = 2\pi \frac{W_{so}}{W_{do}} = \frac{2\pi L I_0^2}{2\pi I_0^2 R/\omega_0}$

$$\text{or} \quad \boxed{Q_{os} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}}.$$

voltage magnification: Voltage magnification is defined as the ratio of voltage across inductor (V_L) or voltage across capacitor (V_C) to the supply voltage at res.

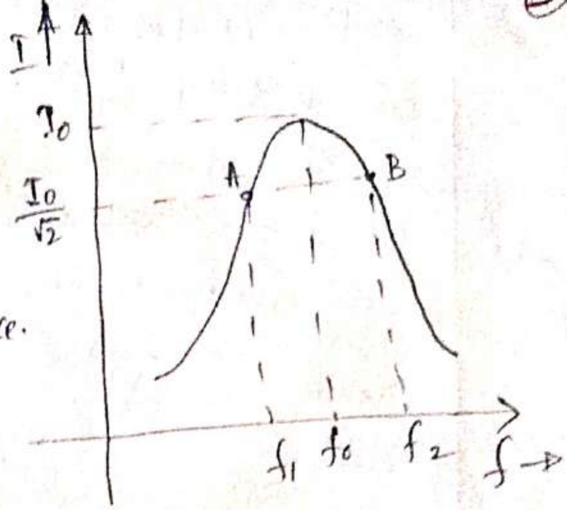
$$\boxed{\text{volt. mag.} = \frac{V_{LO}}{V} = \frac{V_{CO}}{V}}$$

at resonance $V = V_{Ro}$ $\therefore \frac{V_{LO}}{V} = \frac{V_{LO}}{V_{Ro}} = \frac{X_{LO}\cdot I_0}{R\cdot I_0} = \frac{X_{LO}}{R} = \frac{\omega_0 L}{R} = Q_0$

and $\frac{V_{CO}}{V} = \frac{V_{CO}}{V_{RO}} = \frac{X_{CO}\cdot I_0}{R I_0} = \frac{X_{CO}}{R} = \frac{X_{LO}}{R} = \frac{\omega_0 L}{R} = Q_0$.

Hence in a series resonant ckt, Q factor is a measure of voltage magnification. now $\boxed{Q_{os} = \frac{\omega_0 L}{R} = \frac{1}{\sqrt{LC}}\times\frac{L}{R} = \frac{1}{R}\sqrt{L/C}}$

(5) Bandwidth (B.W.): Bandwidth of a series resonant ckt is defined as the band of frequencies between two points on either side of resonant frequency where the current falls to $\frac{1}{\sqrt{2}}$ of its maximum value at resonance.



$BW = \Delta f = (f_2 - f_1)$ Hz

$BW = \Delta\omega = (\omega_2 - \omega_1)$ rad/sec.

where f_1 & f_2 are frequencies at A (lower cutoff frequency) and B (at higher cutoff frequency) respectively.

Now power dissipated in the ckt, at $f_1 = P_A = I_1^2 \cdot R = \left(\frac{I_0}{\sqrt{2}}\right)^2 \cdot R = \frac{I_0^2 \cdot R}{2}$

Similarly power dissipate at $f_2 = P_B = I_2^2 \cdot R = \left(\frac{I_0}{\sqrt{2}}\right)^2 \cdot R = \frac{I_0^2 \cdot R}{2}$.

power dissipated resonance $f_0 = P_0 = I_0^2 \cdot R$.

$\therefore \boxed{P_A = P_B = \frac{P_0}{2}}$ $\therefore$ points A & B are called half power points, as at these points power is half of maximum value.

calculating power in decibel (dB) –

$$= 10 \log_{10} \frac{\text{Power at half power points}}{\text{power at resonance}} = 10 \log_{10} \frac{P_0/2}{P_0}$$

$$= 10 \log_{10}(1/2) = -10 \log_{10}(2) = -10 \times 0.301 = -3 \text{ dB}$$

So, points A & B are also called 3dB points, as the power at these points are 3dB below the maximum power at resonance. Hence bandwidth (BW) can be defined as the band of frequencies which lies between lower and upper cutoff (half power) frequencies

B.W. in terms of ckt parameters –

current in series RLC ckt $I = \frac{V}{Z} = \frac{V}{R + j(X_L - X_C)} = \frac{V}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$

$\therefore |I| = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$ —(1)

at half power points, $|I| = I_0/\sqrt{2}$ since $|I_0| = \frac{V}{R}$ then $|I| = \frac{V}{\sqrt{2}R}$ —(2)

Equating (1) & (2) –

⑥ $\frac{V}{\sqrt{2}R} = \frac{V}{\sqrt{R^2+(\omega L - 1/\omega C)^2}}$ or, $\sqrt{R^2+(\omega L-1/\omega C)^2} = \sqrt{2}R$ ⓐ

or, $R^2+(\omega L - \frac{1}{\omega C})^2 = 2R^2$ or, $(\omega L - 1/\omega C)^2 = R^2$ or $(\omega L - \frac{1}{\omega C}) = \pm R$ — ③

Eqn ③ shows that the magnitude of reactive component of Z is equal to resistive component of z, i.e, $Z = R \pm jR = R(1\pm j)$ — ④

Let ω_1 & ω_2 be the lower and upper half power frequencies — then from eqn ③ $\omega_1 L - 1/\omega_1 C = -R$ (reactance at ω_1) (−ve sign indicates that below resonance cap. reactance $1/\omega C$ is greater than ind. reactance ωL)

$\Rightarrow \omega_1^2 LC - 1 + \omega_1 CR = 0$

$\Rightarrow \omega_1^2 + \frac{R}{L}\omega_1 - \frac{1}{LC} = 0$ — ⑤

This is a quadratic eqn of ω_1 whose two roots are given by —

$$\omega_1 = -\frac{R}{2L} \pm \sqrt{(R/2L)^2 + \frac{1}{LC}} \quad — ⑥$$

$ax^2+bx+c=0$, has roots $\alpha_1, \alpha_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

−ve value of ω_1 is meaningless, so, $\Rightarrow \omega_1 = -\frac{R}{2L} + \sqrt{(\frac{R}{2L})^2+\frac{1}{LC}}$.

Let $R/2L = \alpha$, and $\omega_0^2 = \frac{1}{LC}$, then $\boxed{\omega_1 = -\alpha + \sqrt{\alpha^2+\omega_0^2}}$ — ⑦

Similarly, reactance at upper half power frequency ω_2 from eqn ③ —

$\omega_2 L - 1/\omega_2 C = R$ or $\omega_2^2 - R/L\,\omega_2 - \frac{1}{LC} = 0$

which will give the two values of ω_2 as $\omega_2 = \frac{R}{2L} \pm \sqrt{(R/2L)^2 + \frac{1}{LC}}$

now neglecting the −ve value of ω_2 and putting $R/2L = \alpha$ and $\omega_0^2 = \frac{1}{LC}$

$$\boxed{\omega_2 = \alpha + \sqrt{\alpha^2+\omega_0^2}} \quad — ⑧.$$

now subtracting eqn ⑦ from eqn ⑧, $\boxed{\omega_2 - \omega_1 = \Delta\omega = 2\alpha = + R/L = BW}$ — ⑨

Eqn ⑨ shows that BW of a series RLC ckt is dependant on R/L ratio, but independant of C, but ω_0 depends upon C.

now multiplying eqns ⑦ × ⑧, $\omega_1 \cdot \omega_2 = (-\alpha + \sqrt{\alpha^2+\omega_0^2})(\alpha + \sqrt{\alpha^2+\omega_0^2})$

$= \alpha^2 - \alpha^2 + \omega_0^2 = \omega_0^2$

or $\boxed{\omega_0 = \sqrt{\omega_1 \omega_2}}$ — ⑩

Eqn ⑩ shows that resonant freq. is the (G.M.) geometric mean of the lower & upper half power frequencies.

⑥ Relation between B.W. ($\Delta\omega$), resonant freq (ω_0) and quality factor (Q) of a series RLC ckt: for a series RLC ckt $Z = R + j(\omega L - 1/\omega C) = R[1 + j\{\frac{\omega L}{R} - \frac{1}{\omega CR}\}]$

$= R[1 + j\{\frac{\omega_0 L}{R} \times \frac{\omega}{\omega_0} - \frac{1}{\omega_0 CR} \times \frac{\omega_0}{\omega}\}]$, now putting $Q_0 = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$

7

$$Z = R\left[1 + jQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right] = R\left[1 + jQ_0\left(\frac{\omega^2 - \omega_0^2}{\omega\omega_0}\right)\right] = R\left[1 + jQ_0\frac{(\omega-\omega_0)(\omega+\omega_0)}{\omega\cdot\omega_0}\right]$$

at frequencies near resonance $\omega + \omega_0 = 2\omega$,

so $Z = R\left[1 + jQ_0 \times 2\left(\frac{\omega - \omega_0}{\omega_0}\right)\right] = R(1 + j2Q_0\delta)$ — ①

where $\boxed{\delta = \frac{\omega - \omega_0}{\omega_0} = \frac{f - f_0}{f_0}}$ (δ is called the per unit frequency deviation. For $f < f_0$, δ is -ve and for $f > f_0$, δ is +ve, at $f = f_0$, $\delta = 0$)

now at half power frequencies, $Z = R(1 \pm j)$ — ②

at lower half power frequency ω_1, $Z_{\omega_1} = R - jR = \sqrt{2}R\angle{-45^\circ}$ — ③

and at upper half power frequency ω_2, $Z_{\omega_2} = R + jR = \sqrt{2}R\angle 45^\circ$ — ④

now comparing eqns ① & ②, $2Q_0\delta = \pm 1$ or $\boxed{\delta = \pm\frac{1}{2Q_0}}$ — ⑤

in eqn ⑤ +ve sign for δ_2 and -ve sign for f_1 i.e, $\frac{f_2 - f_0}{f_0} = +\frac{1}{2Q_0}$ and $\frac{f_1 - f_0}{f_0} = -\frac{1}{2Q_0}$

subtracting 2nd from 1st, $\frac{f_2 - f_1}{f_0} = \frac{1}{2Q_0} + \frac{1}{2Q_0} = \frac{1}{Q_0}$

or, $f_2 - f_1 = \frac{f_0}{Q_0}$ or $\omega_2 - \omega_1 = \frac{\omega_0}{Q_0}$ or $\boxed{Q_0 = \frac{\omega_0}{\omega_2 - \omega_1}} = \frac{R.F.}{BW.}$

Now, selectivity (S) of a resonant ckt is defined as the ratio of resonant frequency to B.W. at, resonance, $S_0 = \frac{\omega_0}{BW} = \frac{\omega_0}{\Delta\omega}$

for a series RLC ckt, putting $\Delta\omega = R/L$, $\boxed{S_0 = \frac{\omega_0 L}{R} = Q_0}$

Approximate relationship between $\omega_1, \omega_2, \omega_0$ and B.W. $(\Delta\omega)$

we know, $\omega_1 = -R/2L + \sqrt{(R/2L)^2 + \omega_0^2}$, putting $Q_0 = \frac{\omega_0 L}{R}$ or $R/L = \frac{\omega_0}{Q_0}$

$$= -\frac{\omega_0}{2Q_0} + \sqrt{\frac{\omega_0^2}{4Q_0^2} + \omega_0^2} = -\frac{\omega_0}{2Q_0} + \omega_0\sqrt{\frac{1}{4Q_0^2} + 1} \quad \text{— ①}$$

Similarly $\omega_2 = \frac{\omega_0}{2Q_0} + \omega_0\sqrt{\frac{1}{4Q_0^2} + 1}$ — ② $\therefore \boxed{BW = \omega_2 - \omega_1 = \frac{\omega_0}{Q_0} = 2\alpha}$

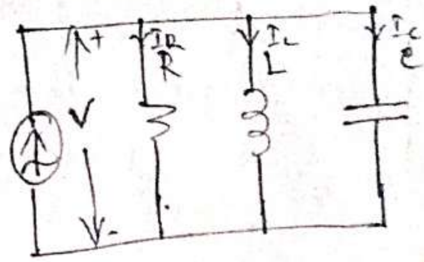
approximations: when $Q_0 \geq 5$, $1 + \frac{1}{4Q_0^2} \simeq 1$ (approximately)

so, $\omega_1 = \omega_0 - \frac{\omega_0}{2Q_0} = \omega_0 - \frac{1}{2}\Delta\omega$ $\therefore \boxed{\omega_1 = \text{resonant freq.} - \tfrac{1}{2}\text{B.W}}$

Similarly $\omega_2 = \omega_0 + \frac{\omega_0}{2Q_0} = \omega_0 + \frac{1}{2}\Delta\omega$ $\therefore \boxed{\omega_2 = \text{resonant freq.} + \tfrac{1}{2}\text{BW}}$

also $\boxed{\omega_0 = \frac{\omega_1 + \omega_2}{2}}$

(7) Resonance in parallel RLC ckt :



In the fig. is shown a parallel RLC ckt connected to a sinusoidal current source I_s.

Net admittance of the ckt is given-

$$Y = Y_R + Y_L + Y_C = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C$$
$$= \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

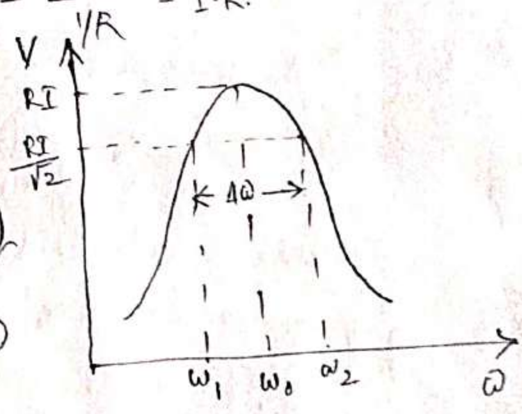
at resonance, imaginary part of $Y = 0$, i.e, at $\omega = \omega_0$, $Im(Y) = 0$.

$\omega_0 C - \frac{1}{\omega_0 L} = 0$, $\omega_0^2 = \frac{1}{LC}$ or $\boxed{\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{1/LC}}$ rad/sec.

or $2\pi f_0 = \sqrt{\frac{1}{LC}}$ or $\boxed{f_0 = \frac{1}{2\pi}\sqrt{1/LC}}$ Hz.

At resonance, $Y_0 = \frac{1}{R}$, which so admittance is minimum at resonance so, voltage is maximum, given by $|V_0| = \frac{|I|}{1/R} = I \cdot R$.

at $\omega = 0$, $Z_L = 0$, so $V = 0$, as $\omega \to \infty$, $Z_C = \frac{1}{j\omega C} \to 0$, again $V \to 0$. In the figure is shown the response curve of a parallel RLC ckt. (variation of $|V|$ vs ω)

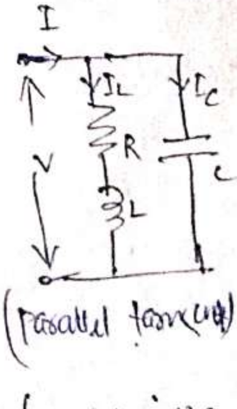


At resonance, parallel RLC ckt behaves as only a resistor. The parallel LC ckt, called as tank ckt, behaves as an open ckt.

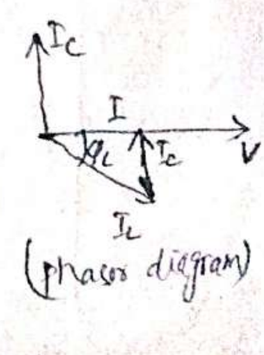
properties of 2nd order parallel RLC ckt

1. Resonant frequency $\omega_0 = \frac{1}{\sqrt{LC}}$ rad/sec, $f_0 = \frac{1}{2\pi\sqrt{LC}}$ cycle/sec.
2. Below resonance, the ckt acts like RL ckt, above resonance as RC ckt, and at resonance the ckt acts like resistor.
3. If the conductance is zero (resistance is infinite), at resonance the ckt acts like an open circuit.
4. Bandwidth $B.W. = \Delta\omega = \omega_2 - \omega_1 = 1/RC$
5. Quality factor $Q_{op} = R\sqrt{C/L}$

(8) Resonance in practical parallel RLC ckt



(parallel tank ckt)



(phasor diagram)

A practical parallel RLC ckt, consists of a coil of resistance $R(\Omega)$, inductance L (Henry) in parallel with a capacitance C (Farad). Now admittance of the ckt-

$$Y = \frac{1}{Z} = \frac{1}{Z_L} + \frac{1}{Z_C} = \frac{1}{R + jX_L} + \frac{1}{-jX_C}$$
$$= \frac{1}{R + j\omega L} + \frac{1}{-j(1/\omega C)} = \frac{1}{R + j\omega L} + j\omega C.$$

$= j\omega C + \frac{R - j\omega L}{R^2 + \omega^2 L^2} = \frac{R}{R^2+\omega^2 L^2} + j\omega\left\{C - \frac{L}{R^2+\omega^2L^2}\right\}$ — (1)

at resonance $Im(Y) = 0$, so, $C - \frac{L}{R^2 + \omega_0^2 L^2} = 0$.

or, $R^2 + \omega_0^2 L^2 = 1/C$ or $\omega_0^2 L^2 = \frac{L}{C} - R^2$ or $\omega_0^2 = \frac{1}{CL} - \frac{R^2}{L^2}$

or, $\boxed{\omega_0 = \sqrt{\frac{1}{LC} - R^2/L^2}}$ or $\boxed{f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}}$ — (2)

if R will be very very small, so that $\frac{R^2}{L^2}$ can be neglected in comparison to $\frac{1}{LC}$

then $\boxed{f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi}\sqrt{1/LC}}$ — (3) same as that in case of series ckt.

<u>phasor diagram</u>: Phasor dgm. for the ckt. is as shown in the figure. The capacitor current I_C leads the supply voltage V by 90° and current in the coil I_L lags behind V by an angle $\phi_L = \tan^{-1}\frac{X_L}{R}$. The supply current I is the phasor sum of I_C & I_L.

<u>Input admittance at resonance (Y_0)</u>. making Imaginary part equal to zero in the expression for Y, we get $\boxed{Y_0 = \frac{R}{R^2+\omega_0^2L^2}}$ — (4)

Thus impedance at resonance $Z_0 = \frac{1}{Y_0}$

$= \frac{R^2 + \omega_0^2 L^2}{R} = R + \frac{\omega_0^2 L^2}{R}$.

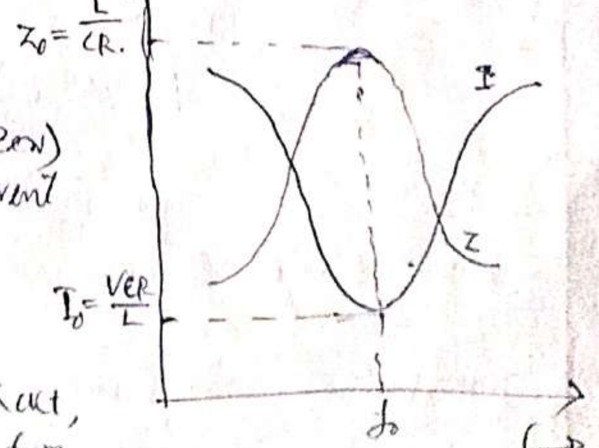
putting $\omega_0^2 L^2 = \frac{L}{C} - R^2$ from eqn (2) $Z_0 = R + \frac{(L/C) - R^2}{R} = R + \frac{L}{CR} - R$

Z_0 is called the <u>dynamic impedance</u> of the ckt. It is pure resistance as it is independent of freq.

or $\boxed{Z_0 = \frac{L}{CR}}$ — (5)

It is seen that lower the value of resistance of the coil, higher is the impedance and so current at resonance is minimum. Hence the parallel resonant ckt is called a <u>rejector circuit</u>. current at resonance is given as $\boxed{I_0 = \frac{V}{Z_0} = \frac{VCR}{L}}$

if R = 0 (resistance of the coil is zero) then the ckt will draw no current at resonance. In the fig. is shown the current Vs freq. and impedance Vs frequency curve of a tank ckt.



<u>current Magnification</u>: In a tank ckt, at resonance the current taken from the supply can be greatly magnified. Current magnification $= \frac{I_C}{I}$

$= \frac{I_L \sin\phi_L}{I_L \cos\phi_L} = \tan\phi_L = \frac{\omega_0 L}{R} =$ Q factor of the ckt at resonance

Q factor is a measure of current magnification in parallel resonant ckt.

(8) Quality factor for a parallel RLC resonant ckt.

Let $v = V_m \sin\omega t$ so at resonance $v = V_m \sin\omega_0 t$ is the cnst. voltage. current in the inductor coil i.e. $i_L = \frac{V_m}{X_{L0}} \sin(\omega_0 t - 90°) = -\frac{V_m}{\omega_0 L}\cos\omega_0 t$

resonant freq. $\omega_0 = \frac{1}{\sqrt{LC}}$

Now instantaneous energy in the ckt is $w(t) = \frac{1}{2}Li^2 + \frac{1}{2}Cv^2$

$$= \frac{1}{2}\frac{LV_m^2\cos^2\omega_0 t}{\omega_0^2 L^2} + \frac{1}{2}CV_m^2\sin^2\omega_0 t = \frac{1}{2}CV_m^2\cos^2\omega_0 t + \frac{1}{2}CV_m^2\sin^2\omega_0 t$$

$$= \frac{1}{2}CV_m^2(\cos^2\omega_0 t + \sin^2\omega_0 t) = \frac{1}{2}CV_m^2 \text{ is a const.} \qquad (1)$$

∴ max. energy stored in the ckt $= \frac{1}{2}CV_m^2$

Average power loss in the ckt $P = \frac{V^2}{R} = \frac{(V_m/\sqrt{2})^2}{R} = \frac{V_m^2}{2R}$

∴ energy lost per cycle $= P \times \frac{2\pi}{\omega_0} = \frac{\pi}{\omega_0}\frac{V_m^2}{R}$ — (2)

∴ Q factor of the parallel RLC ckt at resonance $= \frac{2\pi \times \text{max. energy store in the ckt}}{\text{total energy lost per cycle}}$

$$= \frac{2\pi\left(\frac{1}{2}CV_m^2\right)}{\pi V_m^2/\omega_0 R} = \omega_0 CR \quad \therefore \boxed{Q = \omega_0 CR}$$

Now comparing the expression for Q factor in both series & parallel RLC ckt.

for series RLC ckt $\boxed{Q_{0s} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}}$ and $\boxed{Q_{0P} = \omega_0 CR = \frac{R}{\omega_0 L}}$ for parallel RLC ckt

These expressions corresponds under duality transformation as $R \to 1/R$, $L \to C$, $C \to L$

in both the cases $\boxed{B.W. = \omega_0/Q_0}$

(9) Relation between resonant freq. (ω_0), Bandwidth (BW) & Quality factor (Q_{0P})

For a parallel RLC ckt, $Y = \frac{1}{R} + j\left\{\omega C - \frac{1}{\omega L}\right\}$ — (1)

$$\therefore Z = \frac{1}{\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)} = \frac{R}{1 + j\left(\omega CR - R/\omega L\right)}$$

$$\text{or } \frac{Z}{R} = \frac{1}{1 + j\left(\omega CR - R/\omega L\right)} \qquad (2)$$

Quality factor for a parallel RLC ckt, $Q_{0P} = \omega_0 CR = R/\omega_0 L \Rightarrow CR = \frac{Q_{0P}}{\omega_0}$

and $R/L = Q_{0P}\cdot\omega_0$. also at resonance $Z_0 = R$.

putting the above values in eqn (2)

$$\frac{Z}{Z_0} = \frac{1}{1 + j\left(\frac{\omega}{\omega_0}Q_0 - \frac{\omega_0}{\omega}Q_0\right)} = \frac{1}{1 + jQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \qquad (3)$$

Now band width for the parallel resonant RLC ckt is defined as — the difference between two frequencies at which the ckt impedance has fall on to $1/\sqrt{2}$ of its value at resonance (i.e. Z becomes $Z_0/\sqrt{2}$ or $\left|\frac{Z}{Z_0}\right| = \frac{1}{\sqrt{2}}$

from eqn (3) at half power points $-\ Q_0\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)=\pm 1$ — (4)

let ω_1 & ω_2 are the lower & upper cutoff frequencies, then

$$\frac{\omega_1}{\omega_0}-\frac{\omega_0}{\omega_1}=-\frac{1}{Q_0} \quad (5) \quad \text{and} \quad \frac{\omega_2}{\omega_0}-\frac{\omega_0}{\omega_2}=+\frac{1}{Q_0} \quad (6)$$

from (5) $\frac{\omega_1}{\omega_0}-\frac{\omega_0}{\omega_1}=-\frac{1}{Q_0}$ or $\omega_1^2+\frac{\omega_0}{Q_0}\omega_1-\omega_0^2=0$ — (7)

This is a quadratic eqn in ω_1 and its two values of ω_1 are -

$$\omega_1=\frac{1}{2}\left[-\frac{\omega_0}{Q_0}\pm\sqrt{\frac{\omega_0^2}{Q_0^2}+4\omega_0}\right]=-\frac{1}{2}\frac{\omega_0}{Q_0}\pm\sqrt{\omega_0^2+\frac{\omega_0^2}{4Q_0^2}}$$

negative value of ω_1 is meaningless so $\boxed{\omega_1=-\frac{1}{2}\frac{\omega_0}{Q_0}+\sqrt{\omega_0^2+\frac{\omega_0^2}{4Q_0^2}}}$ (8)

or $\boxed{\omega_1=-\frac{1}{2}\frac{\omega_0}{Q_0}+\omega_0\sqrt{1+\frac{1}{4Q_0^2}}}$ — (8)

Similarly from (6) we can find $\boxed{\omega_2=+\frac{1}{2}\frac{\omega_0}{Q_0}+\omega_0\sqrt{1+\frac{1}{4Q_0^2}}}$ — (9)

* now Subtracting (9) from (8) $\boxed{B.W.=\omega_2-\omega_1=\frac{\omega_0}{Q_0}}$ or $\boxed{Q_0=\frac{\omega_0}{BW}}$

* also multiplying (8) & (9) $\omega_1\cdot\omega_2=\left(\omega_0\sqrt{1+\frac{1}{4Q_0^2}}\right)^2-\left(\frac{1}{2}\frac{\omega_0}{Q_0}\right)^2$

$$=\omega_0^2\left(1+\frac{1}{4Q_0^2}\right)-\frac{\omega_0^2}{4Q_0^2}$$

$$=\omega_0^2$$

$\therefore \boxed{\omega_1\cdot\omega_2=\omega_0^2}$ or $\boxed{\omega_0=\sqrt{\omega_1\cdot\omega_2}}$ (10) resonant freq. is the geometric mean of lower & upper half power frequencies

<u>Approximate expressions</u>: when $Q_0\geq 5$; $\frac{1}{4Q_0^2}\leq\frac{1}{100}$

and $\left(1+\frac{1}{4Q_0^2}\right)=1$ approximately,

putting these values in the exps. for ω_1 & ω_2 in eqns (8) & (9).

$\omega_1=\omega_0-\frac{\omega_0}{2Q_0}$ and $\omega_2=\omega_0+\frac{\omega_0}{2Q_0}$. putting $BW=\frac{\omega_0}{Q_0}=\omega_2-\omega_1$

$=\omega_0-\frac{1}{2}BW$ $\qquad$ $=\omega_0+\frac{1}{2}BW$

now, $\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}=\frac{\omega^2-\omega_0^2}{\omega_0\cdot\omega}=\frac{(\omega-\omega_0)(\omega+\omega_0)}{\omega_0\cdot\omega}=2\left(\frac{\omega-\omega_0}{\omega_0}\right)$ [$\because$ at freq. near resonance $\omega+\omega_0=2\omega$]

now, $\frac{Z}{Z_0}=\frac{1}{1+jQ_0\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)}=\frac{1}{1+jQ_0\times 2\left(\frac{\omega-\omega_0}{\omega_0}\right)}=\frac{1}{1+j2Q_0\delta}$ — (11)

where $\delta=\frac{\omega-\omega_0}{\omega_0}=\frac{f-f_0}{f_0}$ is the per unit freq. deviation.

At half power frequencies, Z becomes equal to $\left|\frac{Z_0}{\sqrt{2}}\right|$ or $|Z/Z_0|=\frac{1}{\sqrt{2}}$

$\Rightarrow 1+4Q_0^2\delta^2=2$ or $4Q_0^2\delta^2=1$, $2Q_0\delta=\pm 1$

or $\boxed{\delta=1/2Q_0}$

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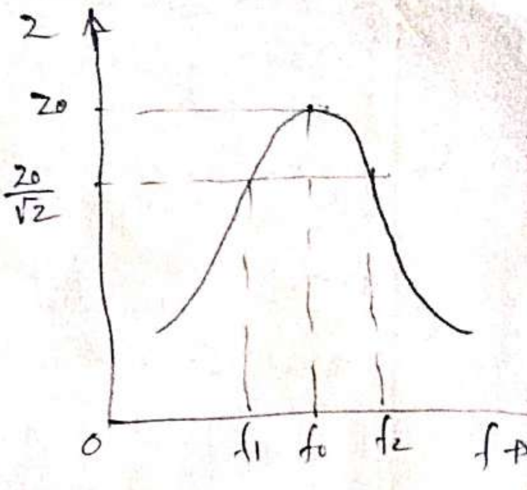
$\therefore$ circuit impedance at ω_1 & ω_2 are -

$$Z(\omega_1) = \frac{Z_0}{1-j1} = \frac{Z_0}{\sqrt{2}} \angle 45^\circ$$

and $$Z(\omega_2) = \frac{Z_0}{1+j1} = \frac{Z_0}{\sqrt{2}} \angle -45^\circ.$$



In the fig. is shown the impedance Vs freq. curve. impedance is maximum at $f_0(\omega_r) = Z_0$. and ~~and~~ at ω_1 & ω_2 it falls to $\frac{1}{\sqrt{2}}$ of Z_0. Below f_0, impedance is inductive and above f_0, impedance is capacitive.

Ch-5 LAPLACE TRANSFORM AND ITS APPLICATIONS. ①

5.1 Laplace Transformation: Definition

Laplace transformation of a time function $f(t)$ is given by $L\, f(t) = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$

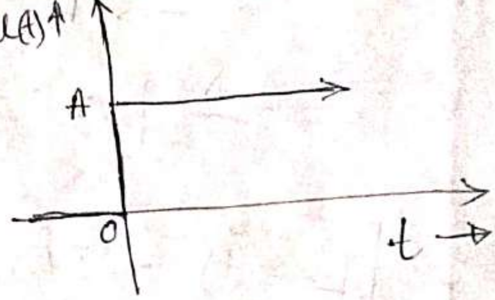
where s is a complex frequency given by $s = \sigma + j\omega$.

$\sigma \rightarrow$ real part and ω is the imaginary part.

Hence by laplace transformation a time function $f(t)$ is transformed to a frequency function $F(s)$.

5.2 Laplace transform of some common functions.

1. Step function $u(t)$



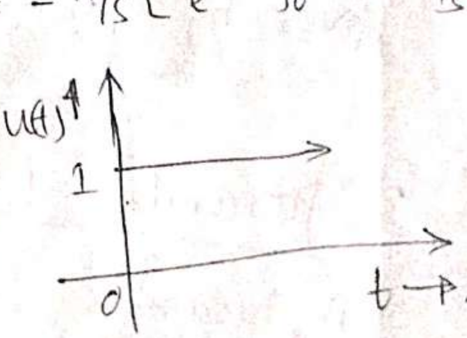
A step function is defined as $u(t) = 0,\ t \leq 0$
$= A,\ t > 0.$

Its laplace transform is given by

$$L[u(t)] = \int_0^\infty u(t) \cdot e^{-st} \cdot dt = \int_0^\infty A \cdot e^{-st} \cdot dt$$

$$= -\frac{A}{s} \int_0^\infty e^{-st} (-s)\, dt = -A/s \left[e^{-st} \right]_0^\infty = -A/s\,(0-1)$$

or, $\boxed{L[u(t)] = A/s}$.

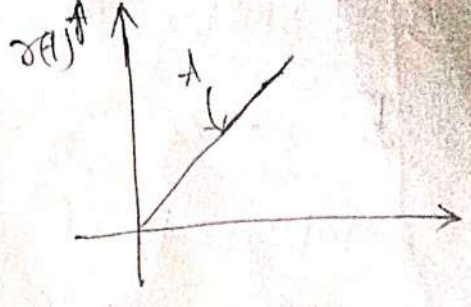


If $A = 1$, then $u(t) = 0,\ t \leq 0$
$= 1\ \ t > 0$

is called unit step function and its laplace transform is $\boxed{L[u(t)] = 1/s}$

2. Ramp function :

A ramp function is given by



$r(t) = 0, \ t \le 0$

$\quad = At, \ t > 0.$

Its laplace transform is $L[r(t)] = \int_0^\infty At \cdot e^{-st} \cdot dt$

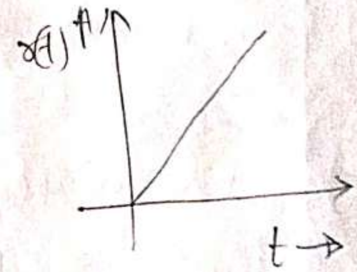
$= A \int_0^\infty t \cdot e^{-st} \cdot dt$

[Applying the formulae of integration by parts i.e, $\int u \cdot dv = u \cdot v - \int v du$
Here $u = t, du = dt, dv = e^{-st} dt$
so, $v = \frac{e^{-st}}{-s}$.

$= t \cdot \frac{e^{-st}}{-s} \Big|_0^\infty - A \int_0^\infty \frac{e^{-st}}{-s} \cdot dt = \frac{A}{s} \int_0^\infty e^{-st} \cdot dt = \frac{A}{s} \left| \frac{e^{-st}}{-s} \right|_0^\infty$

$= -\frac{A}{s^2} \left[e^{-\infty} - e^0 \right] = -\frac{A}{s^2} \cdot (0 - 1) = \frac{A}{s^2}$.

$\therefore \boxed{L \, r(t) = A/s^2}$

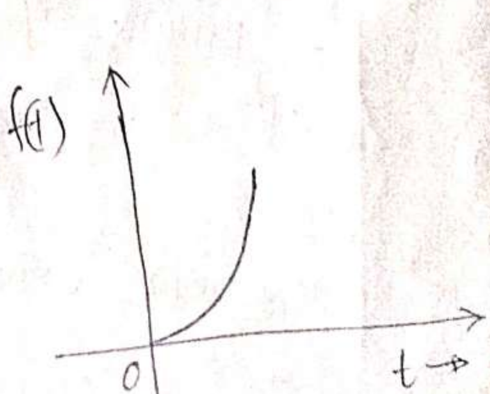


If $A = 1$, then $r(t) = 0, \ t \le 0$.

$\quad = t, \ t > 0$

and is called the unit ramp function, whose laplace is given by $\boxed{L[r(t)] = 1/s^2}$

3. Parabolic function.



A parabolic function is given by

$f(t) = 0, \ t \le 0$

$\quad = At^2, \ t > 0.$

Laplace transform of parabolic function is given by $L[f(t)] = \int_0^\infty At^2 \cdot e^{-st} \cdot dt = A \int_0^\infty t^2 \cdot e^{-st} \cdot dt$

Integrating by parts — $u = t^2,\ dv = e^{-st}\,dt,\ du = 2t\,dt,\ v = \frac{e^{-st}}{-s}$

$$= At^2 \cdot \frac{e^{-st}}{-s}\Big|_0^\infty - 2A\int_0^\infty \frac{e^{-st}}{-s}\, t\, dt = \frac{2A}{s}\int_0^\infty t\, e^{-st}\, dt$$

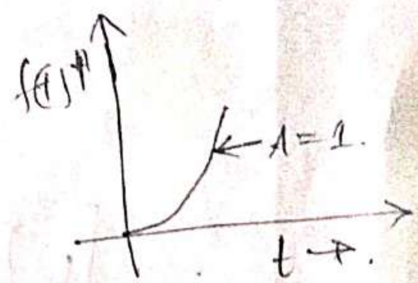
again integration by parts —

$u = t,\ du = dt,\ dv = e^{-st}\,dt,\ v = \frac{e^{-st}}{-s}$.

$$= \frac{2A}{s} \cdot t\, \frac{e^{-st}}{-s}\Big|_0^\infty - \frac{2A}{s}\int_0^\infty \frac{e^{-st}}{-s}\, dt = \frac{2A}{s^2}\left|\frac{e^{-st}}{-s}\right|_0^\infty$$

$$= -\frac{2A}{s^3}(0-1) = \frac{2A}{s^3}$$

$$\therefore \boxed{L\, f(t) = \frac{2A}{s^3}}$$



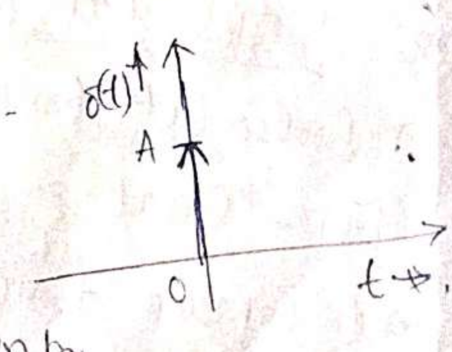
If $A = 1$, then $f(t)$ is called unit parabolic function, $f(t) = 0,\ t < 0$

$= t^2\ \ t > 0$

and its laplace transform is $\boxed{L\, f(t) = \frac{2}{s^3}}$.

* In general for a function $f(t) = t^n$. its laplace transform is given by $\boxed{L[t^n] = \frac{n!}{s^{n+1}}}$

4. Impulse function $\delta(t)$.



An impulse function is defined as -

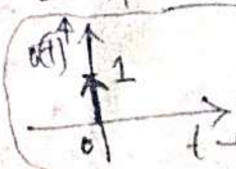
$\delta(t) = A,\ t = 0$

$= 0$ elsewhere

Its laplace transform is given by -

$$L[\delta(t)] = \int_0^\infty A \cdot e^{-st}\, dt = A\left|e^{-st}\right|_{t=0} = A.$$

$$\therefore \boxed{L[\delta(t)] = A}$$

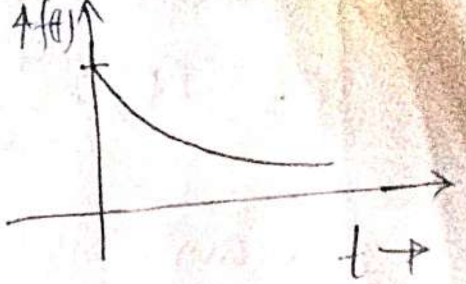


If $A = 1$, then the function is called unit impulse function and its L.T. is $\boxed{L\, \delta(t) = 1}$

5. Exponential function (e^{at})

An exponential function is given by

$f(t) = e^{-at}, \quad t > 0$



Its Laplace transform is -

$$L[f(t)] = \int_0^\infty e^{-at} \cdot e^{-st} \cdot dt = \int_0^\infty e^{-(s+a)t} \cdot dt$$

$$= \frac{e^{-(s+a)t}}{-(s+a)}\Bigg|_0^\infty = -\frac{1}{(s+a)}\left[e^{-\infty} - e^{0}\right]$$

$$= -\frac{1}{(s+a)}(0-1) = \frac{1}{s+a}$$

$$\therefore \boxed{L[e^{-at}] = \frac{1}{s+a}}$$

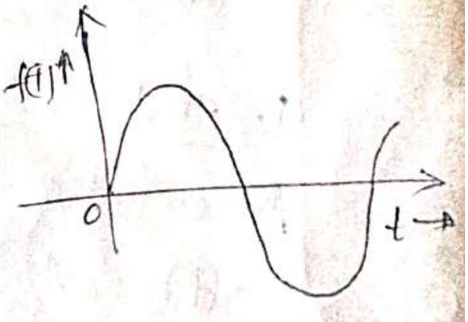
Similarly $\boxed{L[e^{at}] = \frac{1}{s-a}}$

In general, $\boxed{L[e^{\pm at}] = \frac{1}{s \mp a}}$

6. Sinusoidal function ($\sin at$)

A sinusoidal function is given by

$$f(t) = \sin at = \frac{e^{jat} - e^{-jat}}{2j}$$



Its Laplace transform is given by.

$$L[f(t)] = \int_0^\infty \frac{e^{jat} - e^{-jat}}{2j} \cdot e^{-st} \cdot dt$$

$$= \frac{1}{2j}\left[L\{e^{jat}\} - L\{e^{-jat}\}\right] = \frac{1}{2j}\left[\frac{1}{s-ja} - \frac{1}{s+ja}\right]$$

$$= \frac{1}{2j} \times \frac{s+ja-s+ja}{s^2-(ja)^2} = \frac{2ja}{2j(s^2+a^2)} = \frac{a}{s^2+a^2}$$

②

$$\therefore \boxed{L[\sin at] = \frac{a}{s^2+a^2}}$$

Similarly $\boxed{L[\cos at] = \frac{s}{s^2+a^2}}$

7. <u>Laplace transform of derivative of a function.</u>

To find $L[f'(t)] = L\left[\frac{d}{dt}\{f(t)\}\right]$.

Laplace transform of any function $f(t)$ is given by —

$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$, integrating by parts —

Let $f(t) = u$, $dv = e^{-st}\,dt$

$du = f'(t)$, $v = \frac{e^{-st}}{-s}$.

$$= f(t)\cdot\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\cdot f'(t)\cdot dt$$

$$= -\frac{1}{s}\left[f(\infty)\, e^{-\infty} - f(0)\, e^{0}\right] + \frac{1}{s}\int_0^\infty f'(t)\cdot e^{-st}\cdot dt$$

$$= +\frac{1}{s}\left[f(0)\right] + \frac{1}{s}\int_0^\infty f'(t)\, e^{-st}\cdot dt$$

or. $s F(s) = + f(0) + \underbrace{\int_0^\infty f'(t)\, e^{-st}\cdot dt}_{= L[f'(t)]}$

or $\boxed{L[f'(t)] = s F(s) - f(0^+)}$

8. <u>Laplace transform of integration of a function</u>

To find $L[\int f(t)dt]$

we know $F(s) = L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\cdot dt$

integrating by parts — Let $u = e^{-st}$, $du = -s e^{-st} dt$

$dv = f(t)dt$ $v = \int f(t)\, dt$

Similarly $L\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$

In general $L[f^n(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{n-1}(0)$

$$F(s) = e^{-st} \int f(t)dt \Big|_0^\infty - \int_0^\infty \int f(t)dt \, (-s) e^{-st} . dt$$

$$= 0 - \left[\int f(t)\, dt \right]_{t=0} + s \int_0^\infty \left[\int f(t)\, dt \right] e^{-st} . dt$$

$$= -f^{-1}(0) + s . L\left[\int f(t) dt \right]$$

or $$\boxed{L\left[\int f(t)\, dt \right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s}}$$

9. <u>Time shifting ~~function~~ property:</u>

If $L[f(t)] = F(s)$ then $L\, f(t-a) = e^{-as} . F(s)$

$$L[f(t-a)] = \int_0^\infty f(t-a) e^{-st} . dt$$

$$= \int_0^\infty e^{-as} e^{as} f(t-a) e^{-st} . dt$$

$$= e^{-as} \int_0^\infty f(t-a) e^{-s(t-a)} . dt$$

Let $t-a = \tau$, $dt = d\tau$

and as $t \to 0, \tau \to -a$, and as $t \to \infty, \tau \to \infty$.

$$\therefore L[f(t-a)] = e^{-as} \int_{-a}^\infty f(\tau) e^{-s\tau} . d\tau .$$

Since $f(\tau) = 0$ for $\tau < 0$, the lower limit in the above integration can be put to 0.

$$\therefore L[f(t-a)] = e^{-as} \int_0^\infty f(\tau) e^{-s(\tau)} d\tau = e^{-as} . F(s)$$

$\therefore \boxed{L[f(t-a)] = e^{-as} . F(s)}$ In general $\boxed{L[f(t \mp a)] = e^{\pm as} F(s)}$

(10) frequency shifting ~~function~~ property:

If $L[f(t)] = F(s)$ then, $L[e^{-at} f(t)] = F(s+a)$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-at}.\, f(t)\, e^{-st}.\,dt$$

$$= \int_0^\infty f(t)\, e^{-(s+a)t}.\,dt = F(s+a)$$

$\boxed{L[e^{-at} f(t)] = F(s+a)}$ In general $\boxed{L[e^{\mp at} f(t)] = F(s\pm a)}$

(11) Time scaling property:

If $L[f(t)] = F(s)$ then $L[f(t/a)] = a\, F(sa)$

Now, $L[f(t/a)] = \int_0^\infty f(t/a)\, e^{-st}.\,dt$

$$= a \int_0^\infty f(t/a)\, e^{(-as).t/a} . \frac{dt}{a} = a \int_0^\infty f(t/a)\, e^{-as(t/a)}\, d(t/a)$$

Let $t/a = T$,

$$\therefore L[f(t/a)] = a \int_0^\infty f(T)\, e^{-a(sT)} .\, dT.$$

$$\therefore \boxed{L[f(t/a)] = a.\, F(sa)}$$

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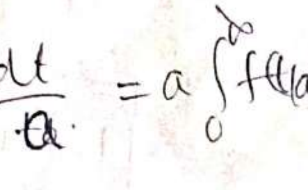
(12) Laplace transform of shifted Time functions.

a) Shifted unit step function:



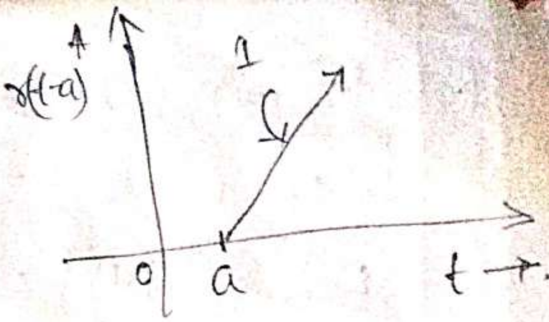
$u(t-a) = 0,\ t \le a$

$\qquad = 1,\ t > a.$

By time shifting property, laplace transform of $u(t-a)$ is given by $L[u(t-a)] = e^{-as} L[u(t)] = e^{-as}. 1/s$

$$\therefore \boxed{L[u(t-a)] = e^{-as}. \frac{1}{s}}$$

b) Shifted unit ramp function.



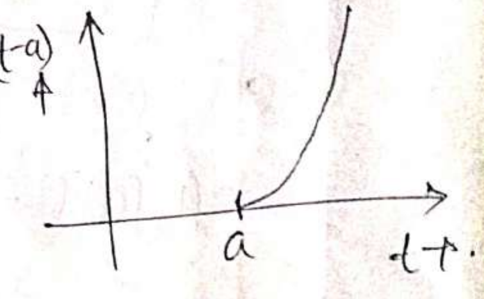
$r(t-a) = 0, \; t \le a$

$\quad\quad = t, \; t > a$

$L[r(t-a)] = e^{-as} \cdot L[r(t)]$

$\therefore \boxed{L[r(t-a)] = e^{-as} \cdot \frac{1}{s^2}}$

c) Shifted unit parabolic function.



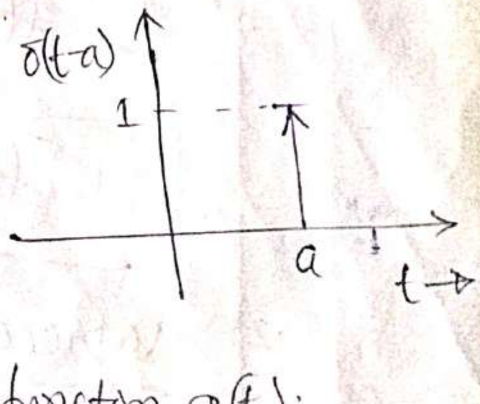
$f(t-a) = 0, \; t \le a$

$\quad\quad = t^2, \; t > a$

$L[f(t-a)] = e^{-as} \cdot L[f(t)] = e^{-as} \cdot L[t^2] = e^{-as} \cdot \frac{2}{s^3}$

$\therefore \boxed{L[(t-a)^2] = \frac{2 \cdot e^{-as}}{s^3}}$

d) Shifted unit impulse function.



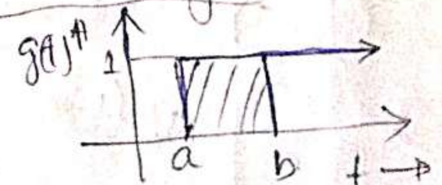
$\delta(t-a) = 0$, elsewhere

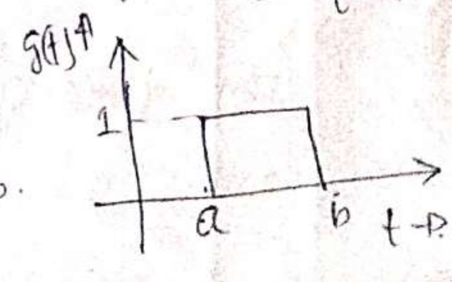
$\quad\quad = 1$, at $t = a$.

$L[\delta(t-a)] = e^{-as} \cdot [L\,\delta(t)] = e^{-as}$

(13) Laplace transformation of a gate function $g(t)$.

A gate function is defined as the super composition of two shifted step functions as shown in the figure.





$u(t-a) = 1, \; t > a$ $\quad\quad$ $u(t-b) = 1, \; t > b$

$\quad\quad = 0, \; t \le a,$ $\quad\quad\quad\quad = 0, \; t \le b.$

The gate function $g(t)$ is defined as -

$$g(t) = u(t-a) - u(t-b) = 1, \quad a < t < b.$$
$$= 0, \text{ outside}$$

Now
$$L[g(t)] = L[u(t-a) - u(t-b)]$$
$$= L\{u(t-a)\} - L\{u(t-b)\}$$
$$= e^{-as} \cdot L[u(t)] - e^{-bs} \cdot L[u(t)]$$
$$= e^{-as} \cdot \frac{1}{s} - e^{-bs} \cdot \frac{1}{s}$$
$$= \frac{1}{s}\left[e^{-as} - e^{-bs}\right].$$

5.3. Inverse laplace Transform

If $L[f(t)] = F(s)$ then $L^{-1}[F(s)] = f(t)$.

$L^{-1}[F(s)]$ is called the inverse laplace transform of the frequency function $F(s)$ which transforms $F(s)$ to $f(t)$, and it is given by —

$$\boxed{f(t) = L^{-1} F(s) = \frac{1}{2\pi j} \int_{\delta_0 - j\omega}^{\delta_0 + j\omega} F(s)\, e^{st} \cdot ds.}$$

usually inv. laplace transform is obtained directly by expanding $F(s) = \frac{M(s)}{N(s)}$ in Heaviside's partial fraction expansion method.

Heaviside's partial fraction expansion method:

1) when $F(s) = \frac{M(s)}{N(s)}$ and $N(s) = (s-x)(s-y)(s-z)$

i.e, the roots x, y, z are real & simple then $F(s) = \frac{A}{s-x} + \frac{B}{s-y} + \frac{C}{s-z}$.

where $A = F(s)(s-x)\big|_{s=x}$, $B = F(s)(s-y)\big|_{s=y}$.

and $C = F(s)(s-z)\big|_{s=z}$.

(ii) when $F(s) = \frac{M(s)}{N(s)}$, $N(s) = (s-x)^n \cdot N_1(s)$

i.e, the roots are real and multiple.

$$F(s) = \frac{A_0}{(s-x)^n} + \frac{A_1}{(s-x)^{n-1}} + \cdots + \frac{A_{n-1}}{(s-x)} + \frac{M_1(s)}{N_1(s)}$$

where $\frac{M_1(s)}{N_1(s)}$ represent the other term of expansion.

and $A_0 = (s-x)^n \cdot F(s)\big|_{s=x}$, $A_1 = \frac{d}{ds} F(s)\big|_{s=x}$.

$A_2 = \frac{1}{\underline{|2}} \cdot \frac{d^2}{ds^2} F(s)\big|_{s=x}$, $\cdots$ $A_n = \frac{1}{\underline{|n}} \frac{d^n}{ds^n} F(s)\big|_{s=x}$.

(iii) when $F(s) = \frac{M(s)}{N_1(s)[s-a+jb][s-a-jb]}$, roots being complex.

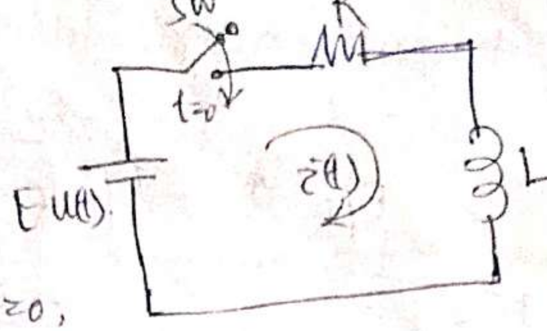
$$F(s) = \frac{A}{s-a-jb} + \frac{B}{s-a+jb} + \frac{M_1(s)}{N_1(s)}$$ where $\frac{M_1(s)}{N(s)}$ represent other terms.

Here $A = \frac{M(a+jb)}{N_1(a+jB)(+2jb)}$, $B = \frac{M(a-jb)}{N_1(a-jB)(-2jb)}$.

(usualy $A = B^*$ or $B = A^*$, $* \rightarrow$ complex conjugate)

5.4. (a) Step Response of series RL circuit:

Step response of a series RL ckt is the response (current) to a step voltage (voltage is a step function) applied, as shown in the fig.



Applying KVL to the ckt —(at time t=0, the switch is closed)

$$E\,u(t) = R\,i(t) + L\cdot\frac{d}{dt}i(t) \quad \text{— (1)}$$

Taking Laplace transform of eqn (1)

$$\frac{E}{s} = R\cdot I(s) + L\cdot[s\,I(s) - i(0^{+})] \quad \text{— (2)}$$

Assuming the inductor was initialy uncharged i.e, $i(0)=0$, eqn (2) reduces to —

$$\frac{E}{s} = R\,I(s) + L\,s\,I(s) = [R+Ls]\,I(s).$$

$$\text{or. } I(s) = \frac{E}{s(R+Ls)} = \frac{E/L}{s\cdot(s+R/L)} \quad \text{— (3)}$$

The response in time domain i.e, $i(t)$ can be found out by ~~at~~ taking inverse L.T. of eqn (3) expanding the RHS of eqn (3) by Heavside's partial fraction—

$$I(s) = \frac{A}{s} + \frac{B}{s+R/L}$$

$$\text{now}\quad A = s\cdot I(s)\big|_{s=0} = \frac{E/L}{s+R/L}\bigg|_{s=0} = \frac{E}{R}$$

$$\text{and}\quad B = (s+R/L)\,I(s)\big|_{s=-R/L} = \frac{E/L}{s}\bigg|_{s=-R/L} = \frac{E/L}{-R/L} = -E/R$$

putting the values of A, B in eqn ③ -

$$I(s) = \frac{E/R}{s} + \frac{-(E/R)}{s+R/L} \quad — ④$$

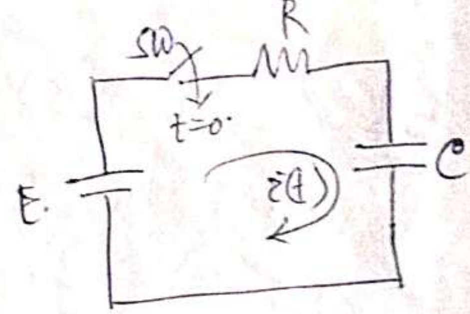
Now taking inverse laplace, we get -

$$i(t) = E/R - E/R\, e^{-\frac{R}{L}t} = \frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)$$

$$\therefore \boxed{i(t) = \frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)}$$

(b) <u>Step response of</u> <u>series RC circuit</u>:

As shown in the fig. the step voltage E is applied to a series R, C circuit at time t = 0 by closing the switch SW



applying KVL to the ckt -

$$E\,u(t) = R\,i(t) + \frac{1}{C}\int i(t)\cdot dt \quad — ①$$

Taking laplace of both sides of eqn ①.

$$\frac{E}{s} = R\,I(s) + \frac{1}{C}\cdot\frac{1}{s}\left[I(s) + i(0)\right]$$

where $i(0) = \int_{-\infty}^{0} i\cdot dt = Q_0 \rightarrow$ current due to initial charge stored in the capacitor

assuming $i(0) = 0$, the capacitor was unenergised -

$$\frac{E}{s} = R\,I(s) + \frac{1}{Cs}I(s) = \left(R+\frac{1}{Cs}\right)I(s).$$

$$\text{or} \quad I(s) = \frac{E}{s\left(R+\frac{1}{Cs}\right)} = \frac{ECs}{s(RCs+1)} = \frac{ECs}{s(s+1/RC)\cdot RC}$$

$$= \frac{E/R}{s+1/RC} + \frac{\cancel{A}}{s+1/RC}$$

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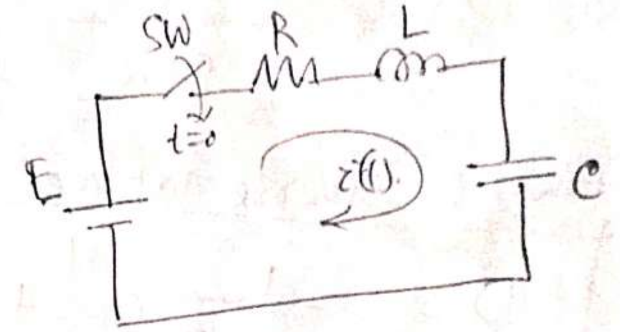
Taking inverse laplace transform -

$$\boxed{i(t) = \frac{E}{R} e^{-\frac{1}{RC}t}}$$

(c) step response of series RLC ckt:

In the fig. is shown a series RLC ckt subjected to a step voltage E at time t=0, when switch is closed.



Applying KVL to the ckt.
assuming the voltage to a unit step voltage and assuming all initial conditions to be zero, applying Laplace transform to the ckt -

$$\frac{1}{s} = \left(R + Ls + \frac{1}{Cs}\right) I(s)$$

$$\text{or} \quad I(s) = \frac{1}{Rs + Ls^2 + 1/C} = \frac{1/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$= \frac{1/L}{(s+\alpha)(s+\beta)}$$

where $\alpha, \beta = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - 1/LC}$

Here, there are 3 possible conditions arises for the solution.

Case-I Both α, β are real & unequal i.e, $\frac{R}{2L} > \frac{1}{\sqrt{LC}}$

$$\therefore I(s) = \frac{1/L}{(s+\alpha)(s+\beta)} = \frac{K_1}{s+\alpha} + \frac{K_2}{s+\beta}.$$

where $K_1 = (s+\alpha) I(s)\Big|_{s=-\alpha} = \frac{1/L}{s+\beta}\Big|_{s=-\alpha} = \frac{1}{L(\beta-\alpha)}$

and $K_2 = (s+\beta) I(s)\Big|_{s=-\beta} = \frac{1/L}{s+\alpha}\Big|_{s=-\beta} = \frac{1}{L(\alpha-\beta)}$

so, $$I(s) = \frac{1/L(\beta-\alpha)}{s+\alpha} + \frac{\frac{1}{L}(\alpha-\beta)}{s+\beta}.$$

taking inverse laplace.

$$\boxed{i(t) = \frac{1}{L(\beta-\alpha)} e^{-\alpha t} + \frac{1}{L(\alpha-\beta)} e^{-\beta t}}$$

Case II. α, β are equal $[\alpha = \beta = \gamma]$ i.e, $\frac{R}{2L} = \frac{1}{\sqrt{LC}}$

$$\therefore I(s) = \frac{1/L}{(s+\gamma)^2}$$

$$\therefore \boxed{i(t) = \frac{1}{L} \cdot t \cdot e^{-\gamma t}}$$

Case-III $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$ that is $\alpha = \beta^*$ (α & β are complex conjugate)

Let $\alpha = -A_0 + jB$, $\beta = -A_0 - jB$

where $|A_0| = \frac{R}{2L}$ and $|B| = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$

using partial fraction expansion -

$$I(s) = \frac{1/L}{(s+A_0-jB)(s+A_0+jB)} = \frac{K}{(s+A_0-jB)} + \frac{K^*}{(s+A_0+jB)}$$

where $K = (S+A_0-jB)\,I(S)\Big|_{S=-A_0+jB}$

$$= \frac{1/L}{(S+A_0+jB)}\Bigg|_{S=-A_0+jB}$$

$$= \frac{1}{j2BL}$$

$$\therefore K^* = -\frac{1}{j2BL}$$

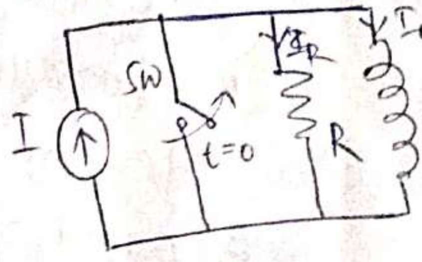
$$\therefore I(S) = \frac{1/j2BL}{S-A_0+jB} - \frac{\frac{1}{j2BL}}{S-A_0-jB}$$

$$\therefore i(t) = \frac{1}{j2BL}\left[e^{-A_0t}\cdot e^{jBt} - e^{-A_0t}\cdot e^{-jBt}\right]$$

$$= \frac{e^{-A_0t}}{BL}\left[\frac{e^{jBt}-e^{-jBt}}{2j}\right] = \frac{e^{-A_0t}}{BL}\sin Bt.$$

(d) Step current response of parallel RL ckt

A step current I is applied to the R, L parallel ckt, at time t=0, by opening the switch sw



As the resistor and inductor are in two parallel path

total current $I = I_R + I_L$ — ①

and $V_R = V_L$ or $R\cdot I_R = L\frac{d}{dt}I_L$ — ②.

multiplying R in both sides of eqn ① we get -

$I\cdot R = R\cdot I_R + R\cdot I_L$ or $I R = R\cdot I_R + L\cdot\frac{d}{dt}I_L$ — ③

taking laplace of eqn ③ with all initial conditions to be zero.

$$R\cdot\frac{I(S)}{S} = R\,I_L(S) + L\,S\,I_L(S) = I_L(S)[R+LS]$$

or, $I_L(s) = I(s)\dfrac{R}{s(R+Ls)} = I(s)\dfrac{R/L}{s(\frac{R}{L}+s)} = I(s)\left[\dfrac{A}{s}+\dfrac{B}{s+R/L}\right]$

where $A = s\cdot I_L(s)\Big|_{s=0} = \dfrac{R/L}{R/L+s}\Big|_{s=0} = 1$

and $B = (s+R/L)\,I_L(s)\Big|_{s=-R/L} = \dfrac{R/L}{-R/L} = -1.$

$\therefore\ I_L(s) = I(s)\left[\dfrac{1}{s} - \dfrac{1}{s+R/L}\right]$

now taking inverse laplace of the above eqn.

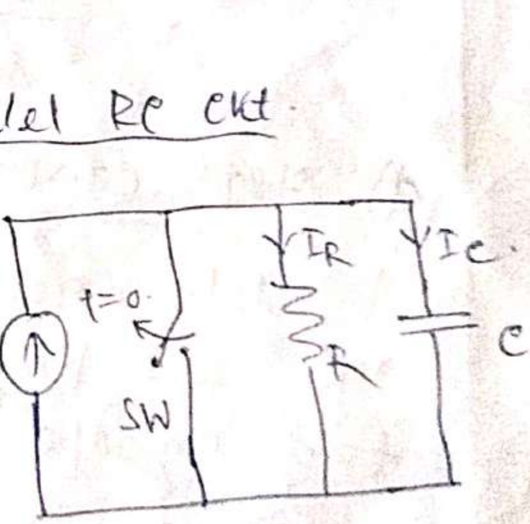
$$\boxed{i_L(t) = i(t)\left[1 - e^{-Rt/L}\right].}$$

and $i_R(t) = i(t) - i_L(t) = i(t) - i(t) + i(t)\,e^{-\frac{R}{L}t} = i(t)\,e^{-\frac{Rt}{L}}$

$$\boxed{i_R(t) = i(t)\,e^{-\frac{R}{L}t}}$$

(e) Step current response of parallel RC ckt.

A step current I is applied to a parallel R, C ckt by opening the switching SW at $t=0$.



we can write –

$I = I_R + I_C$ and $R\cdot I_R = \dfrac{1}{C}\displaystyle\int_0^t I_C\cdot dt$

or $I\cdot R = R\cdot I_R + R I_C = R I_C + \dfrac{1}{C}\displaystyle\int_0^t I_C\cdot dt$

Taking laplace of the above eqn –

$\dfrac{I(s)}{s}\cdot R = R\,I_C(s) + \dfrac{1}{Cs}\{I_C(s)\} = I_C(s)\left[R+\dfrac{1}{Cs}\right]$

or, $I_C(s) = \dfrac{I(s)\cdot R}{s\,(RCs+1)/Cs} = \dfrac{I(s)\cdot R}{s\cdot RC\,(s+1/RC)/Cs} = I(s)\left[\dfrac{1}{s+1/RC}\right]$

Taking inverse laplace transform -

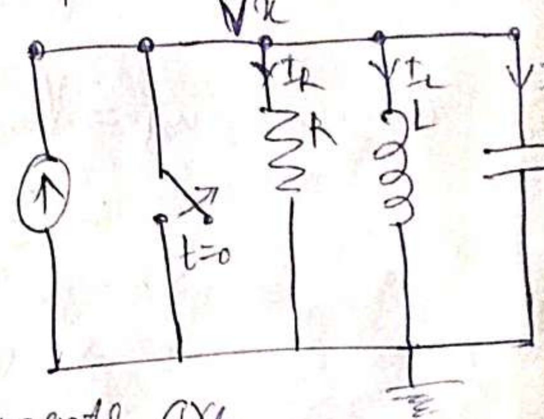
$$\boxed{i_c(t) = i(t)\, e^{-t/RC}}$$

$$i_R(t) = i(t) - i_c(t) = i(t) - i(t)\, e^{-t/RC} = i(t)\left[1 - e^{-t/RC}\right]$$

or $$\boxed{i_R(t) = i(t)\left\{1 - e^{-t/RC}\right\}}$$

(4) <u>Step current response of parallel RLC ckt</u>

A step current I is applied to a parall R, L, C circuit by opening the switch at t=0.



Let V is the voltage at node x and all the elements are initially de energized (all initial conditions = 0)

applying KLC at node x -

$$I = \frac{V}{R} + \frac{1}{L}\int V\cdot dt + C\cdot\frac{dV}{dt} \quad \text{—(1)}$$

Taking laplace transform

$$\frac{I(s)}{s} = \frac{V(s)}{R} + \frac{1}{L}\cdot\frac{1}{s}\left\{V(s)\right\} + Cs\,V(s)$$

$$= V(s)\left[\frac{1}{R} + \frac{1}{Ls} + Cs\right]$$

$$= V(s)\left[\frac{1}{RC}s + \frac{1}{LC} + s^2\right]\frac{C}{s} \quad \text{—(2)}$$

assuming the current source to be a unit step f^n.
i.e, $I(s) = \frac{1}{s}$, the above eqn. reduces to -

$$\frac{1}{s} = V(s)\left(\frac{1}{RC}s + \frac{1}{LC} + s^2\right)\frac{C}{s}$$

or, $V(s) = \dfrac{I/C}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$ —— (3)

In the above exp^{n} putting $\frac{1}{RC} = 2\delta\omega_0$, and $\frac{1}{LC} = \omega_0^2$

eqn (3) reduces to —

$$V(s) = \frac{I/C}{s^2 + 2\delta\omega_0 s + \omega_0^2} = \frac{I/C}{s^2 + 2\delta\omega_0 s + (\delta\omega_0)^2 + \omega_0^2 - (\delta\omega_0)^2}$$

$$= \frac{I}{C\omega_d} \cdot \frac{\omega_d}{(s+\delta\omega_0)^2 + \omega_d^2} \quad \text{—— (4)}$$

where, $\omega_d = \sqrt{\omega_0^2 - \delta(\omega_0)^2} = \sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2}$

= called the damped frequency.

ω_0 = undamped natural frequency.

$\delta \Rightarrow$ damping factor or damping ratio.

Taking inverse laplace transform of eqn (4).

$$\boxed{v(t) = \frac{I}{C\omega_d} e^{-\delta\omega_0 t} \cdot \sin\omega_d t}$$

$$\therefore \boxed{i_R = \frac{v(t)}{R} = \frac{I}{RC\omega_d} \cdot e^{-\delta\omega_0 t} \cdot \sin\omega_d t}$$

and $i_L = \frac{1}{L}\int_0^t v(t)dt = \frac{1}{L}\int_0^t \frac{I}{\omega_d C} e^{-\delta\omega_0 t} \cdot \sin\omega_d t \cdot dt$

$$= \frac{I}{\omega_d LC}\int_0^t e^{-\delta\omega_0 t} \sin\omega_d t \cdot dt$$

$$= \frac{I}{\omega_d LC}\left[\frac{e^{-\delta\omega_0 t}}{\sqrt{(\delta\omega_0)^2 + (\omega_d)^2}} \cdot \sin(\omega_d t - \varphi)\right]$$

where $\varphi = -\tan^{-1}\left(\frac{\omega_d}{\delta\omega_0}\right)$

or, $\boxed{i_L(t) = \frac{1}{\omega_d LC}\left[\frac{e^{-\delta\omega_0 t}}{\sqrt{(\delta\omega_0)^2+(\omega_d)^2}} \cdot \sin\left(\omega_d t + \tan^{-1}\left(\frac{\omega_d}{\delta\omega_0}\right)\right)\right]}$ (7)

Similarly $i_c(t) = C \cdot \frac{d}{dt} v(t)$

Let $\delta\omega_0 = a$, $\omega_d = b$.

$\therefore i_c = C\left(\frac{1}{bC}\right)\left[-a \cdot e^{-at}\sin bt + b e^{-at}\cos bt\right]$

$= \frac{1}{b} e^{-at}\left[b\cos bt - a\sin bt\right]$

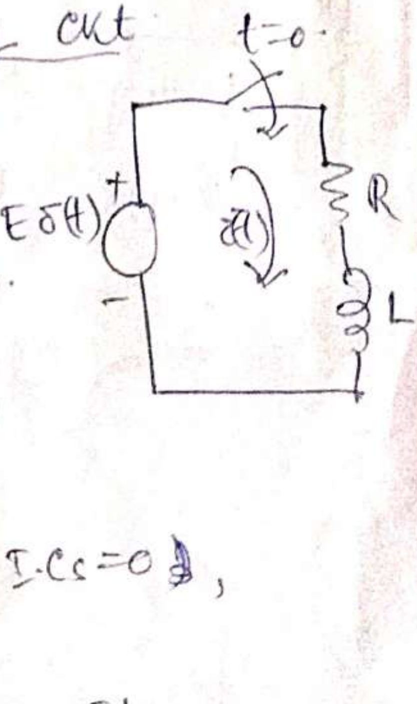
$= \frac{1}{b} e^{-at}\left[\left(\sqrt{a^2+b^2}\right)\sin(bt-\phi)\right]$

where $\phi = \tan^{-1} b/a = \tan^{-1}\frac{\omega_d}{\delta\omega_0}$

$\therefore \boxed{i_c(t) = \frac{e^{-\delta\omega_0 t}}{\omega_d}\left[\sqrt{(\delta\omega_0)^2+\omega_d^2}\right] \cdot \sin\left\{\omega_d t - \tan^{-1}\left(\frac{\omega_d}{\delta\omega_0}\right)\right\}}$

5.5 a) Impulse response of series RL ckt

An impulse voltage of $E\delta(t)$ is applied to a series RL ckt at time $t=0$, by closing the switch.



applying KVL to the ckt -

$E\delta(t) = Ri(t) + L\cdot\frac{d}{dt}i(t)$ —(1)

Taking Laplace transform with all I.Cs = 0,

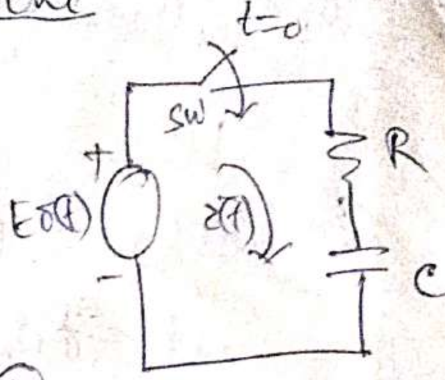
$E = RI(s) + LsI(s)$.

or $I(s) = \frac{E}{R+Ls} = \frac{E}{L[s+R/L]} = \frac{E/L}{s+R/L}$.

taking inverse laplace transform— $\boxed{i(t) = \frac{E}{L}e^{-\frac{Rt}{L}}}$

(b) Impulse response of series RC ckt

An impulse voltage is applied to a series RC ckt, at time t=0, applying KVL to the ckt - (assuming all I.C. =0)



$$E\delta(t) = R\, i(t) + \frac{1}{C}\int i(t)\cdot dt \quad \text{—(1)}$$

taking laplace transform, $E = R I(s) + \frac{1}{Cs} I(s)$

$$\text{or, } I(s) = \frac{E}{R + 1/Cs} = \frac{ECs}{RCs+1} = \frac{ECs}{RC(s + 1/RC)}$$

$$= \frac{E/R \cdot s}{s + 1/RC} = \frac{E}{R}\left[1 - \frac{1/RC}{s + 1/RC}\right].$$

taking inverse laplace,

$$\boxed{i(t) = \frac{E}{R}\left[\delta(t) - \frac{1}{RC} e^{-t/RC}\right]}$$

putting RC = T (time constant).

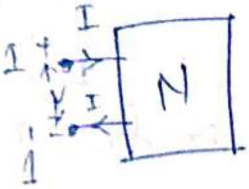
$$\boxed{i(t) = \frac{E}{R}\left[\delta(t) - \frac{1}{T} e^{-t/T}\right]}$$

Ch 6 Network Functions & Parameters ①

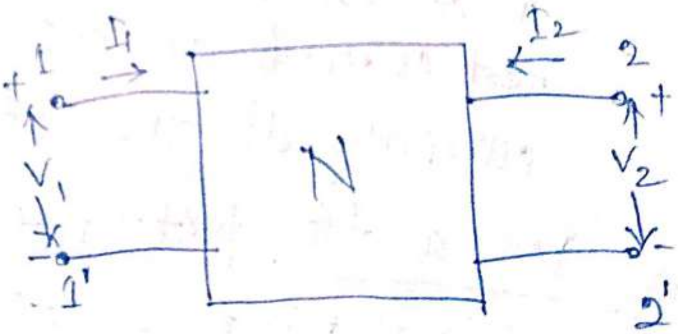
6.1 Network Functions of one port and two port network.

Port: A combination of two terminals i.e, through one terminal, current enters the network and through other, it comes out, is called a port.

One port network: A network having only two terminals or one port is called a one port network



Two port network: Network having two ports or four terminals is called a two port network.



Network Functions: Ratio of Laplace transform of voltages & currents at the ports are called network functions.

Network functions are of two types—

a) Driving point function.

b) Transfer function.

(2) Driving point function: Ratios of laplace transforms of voltages and currents in the same port, are called driving point functions.

For a one port Network:

Driving point impedance function is $Z(s)$

$$\boxed{Z(s) = \frac{V(s)}{I(s)}}$$

and driving point admittance function is $Y(s)$

$$\boxed{Y(s) = \frac{I(s)}{V(s)}}$$

* for a one port network there is no transfer function.

Transfer function

It is the ratio of laplace transform of voltage/ current at one port to that of voltage/ current at the other port.

For a two port network

Driving point impedance function and admittance function at port I are —

$$\boxed{Z_{11}(s) = \frac{V_1(s)}{I_1(s)}} \quad \boxed{Y_{11}(s) = \frac{I_1(s)}{V_1(s)}}$$

Driving point impedance function and admittance function at port II, are -

$$\boxed{Z_{22}(s) = \frac{V_2(s)}{I_2(s)}} \qquad \boxed{Y_{22}(s) = \frac{I_2(s)}{V_2(s)}}$$

voltage transfer function or voltage gain

$$\boxed{V_{12}(s) = \frac{V_2(s)}{V_1(s)}}$$

current transfer function or current gain

$$\boxed{I_{12}(s) = \frac{I_2(s)}{I_1(s)}}$$

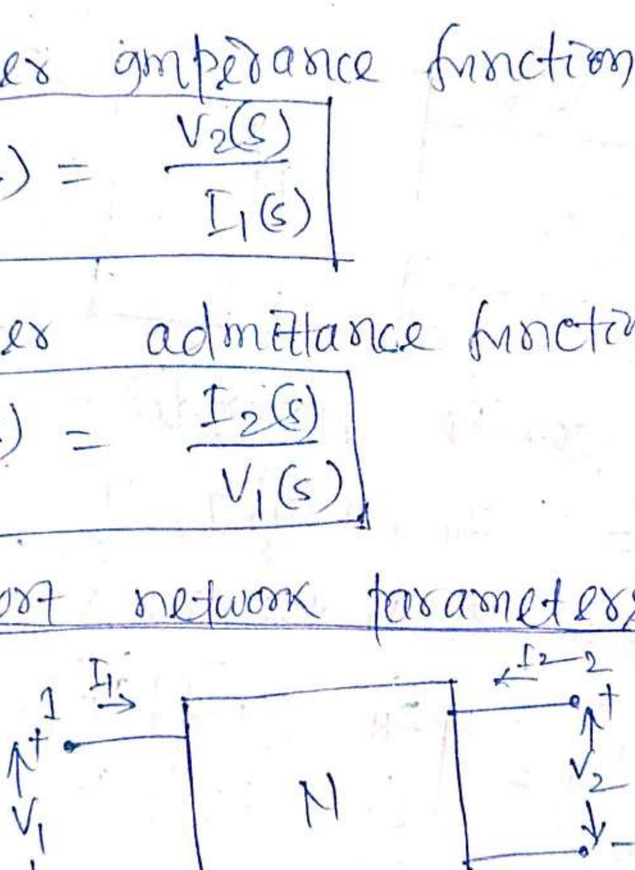
Transfer impedance function -

$$\boxed{Z_{12}(s) = \frac{V_2(s)}{I_1(s)}}$$

Transfer admittance function -

$$\boxed{Y_{12}(s) = \frac{I_2(s)}{V_1(s)}}$$

6.2. Two port network parameters



A two port network is characterised by 4 variables (input voltage V_1, input current I_1, output voltage V_2 & output current I_2). out of these 4, any two can be taken to be independent variables and other two are dependent variables. So out of 4 variables 2 can be chosen to ~~the~~ be

(4) independent (or dependent) in ${}^4C_2$ ways.

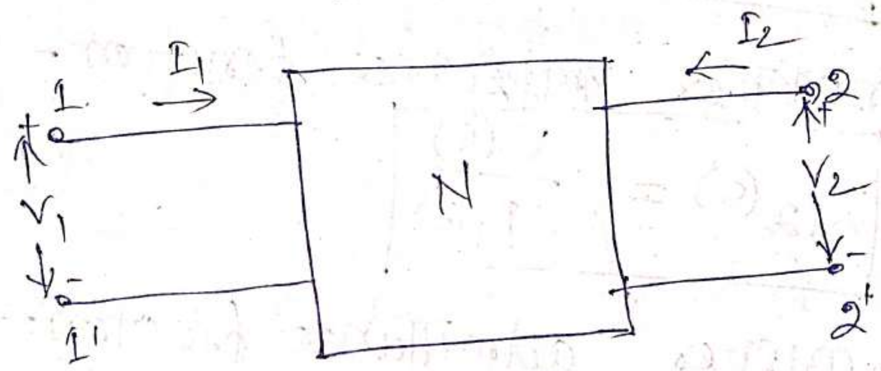
$${}^4C_2 = \frac{\lfloor 4}{\lfloor 2 \lfloor 4-2} = \frac{1\cdot2\cdot3\cdot4}{1\cdot2\cdot1\cdot2} = 6 \text{ ways.}$$

So there are 6 sets of parameters. Z, Y, h, g, ABCD and abcd parameters.

i) <u>Z- parameters</u> : (open circuit or impedance parameters)

V_1 & V_2 are dependent variables and I_1 & I_2 are independent variables.

$$\therefore (V_1, V_2) = f(I_1, I_2) \quad \text{——— (1)}$$



Eqn (1) can be expanded —

$$\left.\begin{aligned} V_1 &= Z_{11} I_1 + Z_{12} I_2 \\ V_2 &= Z_{21} I_1 + Z_{22} I_2 \end{aligned}\right\} \quad \text{——— (2)}$$

$$\text{or, } \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

now, $Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}$, Z_{11} is called input impedance (Z_i) when port II is o.c.

$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$, Z_{12} is called reverse transfer impedance, when port I is o.c. (Z_r)

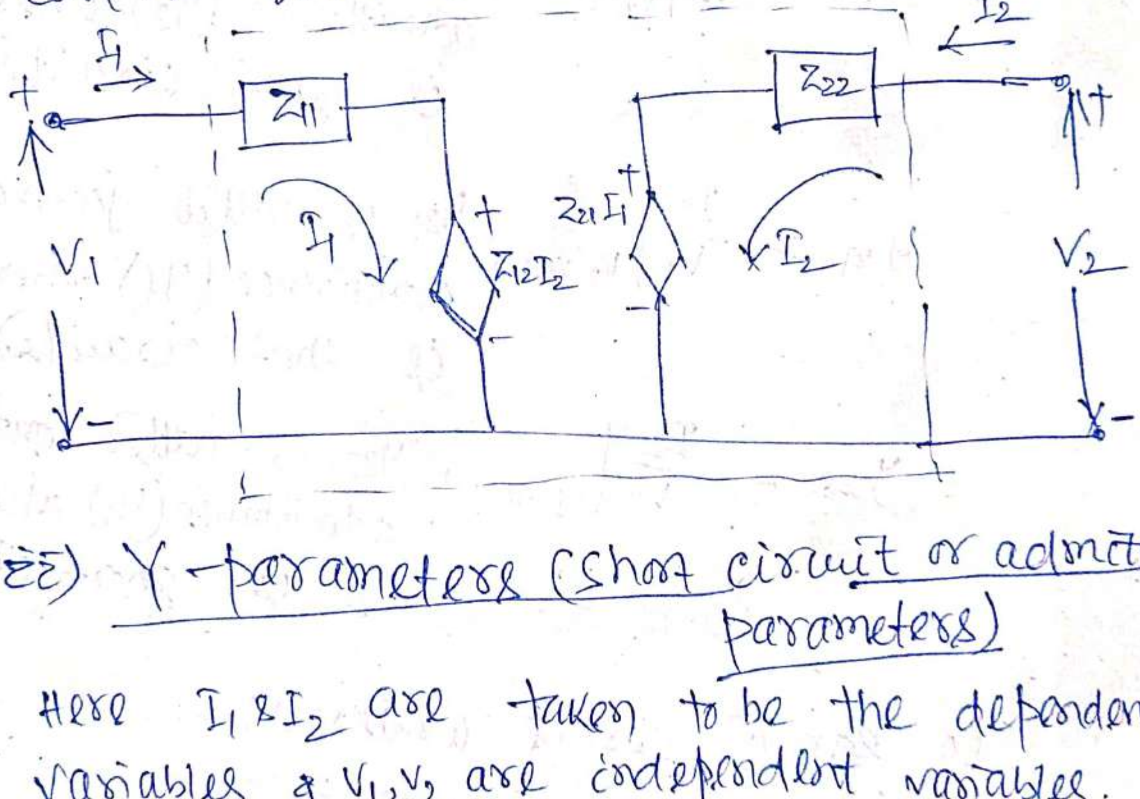
$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$, Z_{21} is called forward transfer impedance (Z_f) when port II is O.C. (5)

$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$, Z_{22} is called output impedance (Z_o), when port I is open ckted.

So, eqn (2) can be represented as —

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Z_i & Z_r \\ Z_f & Z_o \end{pmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

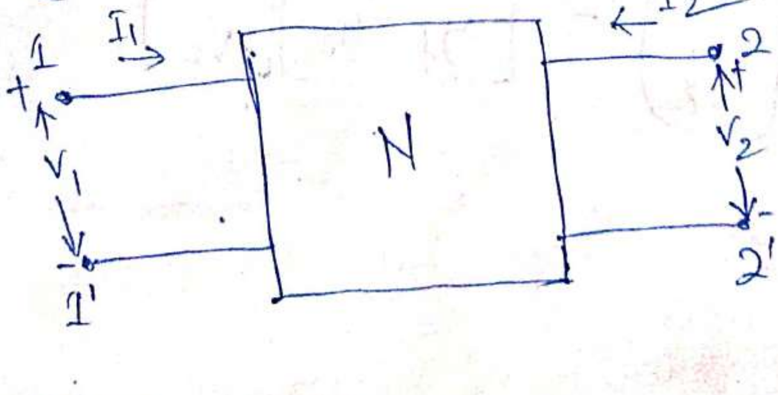
The Z-parameter equivalent circuit (as per eqn (2)) can be given as —



ii) Y-parameters (Short circuit or admittance parameters)

Here I_1 & I_2 are taken to be the dependent variables & V_1, V_2 are independent variables.

$(I_1, I_2) = f(V_1, V_2)$ ——— (1)



(6) eqn (1) can be written as —

$$\left.\begin{aligned} I_1 &= y_{11}V_1 + y_{12}V_2 \\ I_2 &= y_{21}V_1 + y_{22}V_2 \end{aligned}\right\} \quad \text{—— (2)}$$

or, $\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

from eqn (2) —

$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$, y_{11} is called input admittance (y_i) when port two is short circuit

$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$, y_{12} is called reverse transfer admittance (y_r) when port I is short circuited.

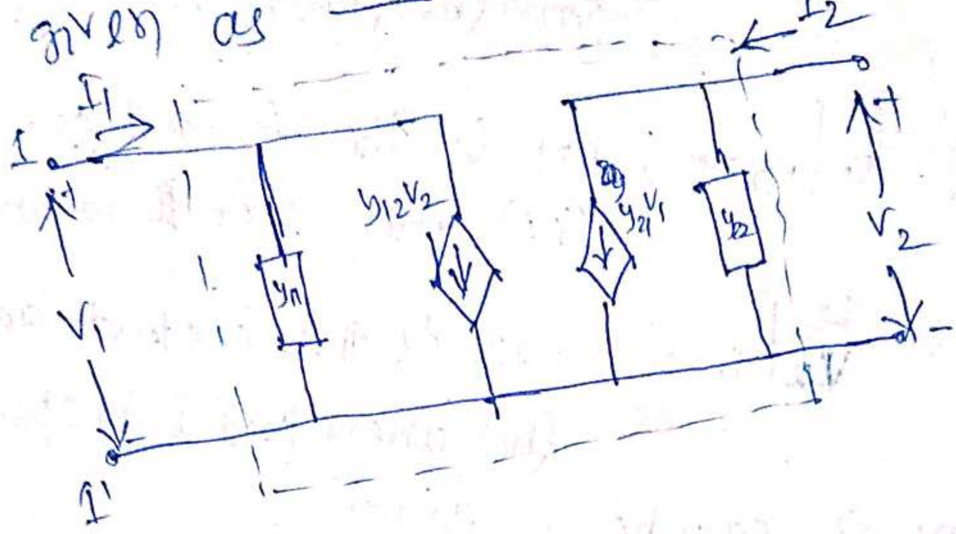
$y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$, y_{21} is called forward transfer admittance (y_f) when port II is short circuited

$y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$, y_{22} is called output admittance (y_o) when port I is short circuited.

So eqn (2) can be written as —

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_i & y_r \\ y_f & y_o \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

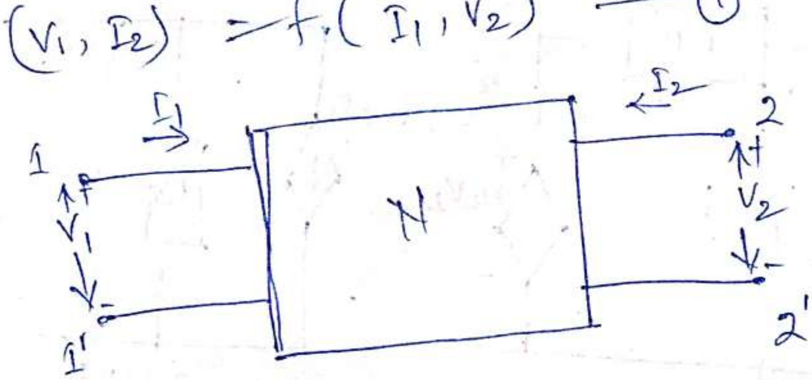
The y-parameter equivalent ciruit can be given as — (7)



Q3. (iii) h parameters (hybrid parameters).

Hybrid parameters are combination of part of z-parameters and part of y-parameters. V_1 & I_2 are taken to be dependent variables and I_1 & V_2 are independent variables.

$(V_1, I_2) = f(I_1, V_2)$ —— (1)



Eqn (1) can be written as —

$V_1 = h_{11} I_1 + h_{12} V_2$

$I_2 = h_{21} I_1 + h_{22} V_2$

or $\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$ —— (2)

from eqns (2) —

$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$, h_{11} is called input impedance h_i, when port II is shortckted.

8

$h_{12} = \frac{V_1}{V_2}\Big|_{I_1=0}$, h_{12} is the reverse volt gain (h_r), when port I is O.C.

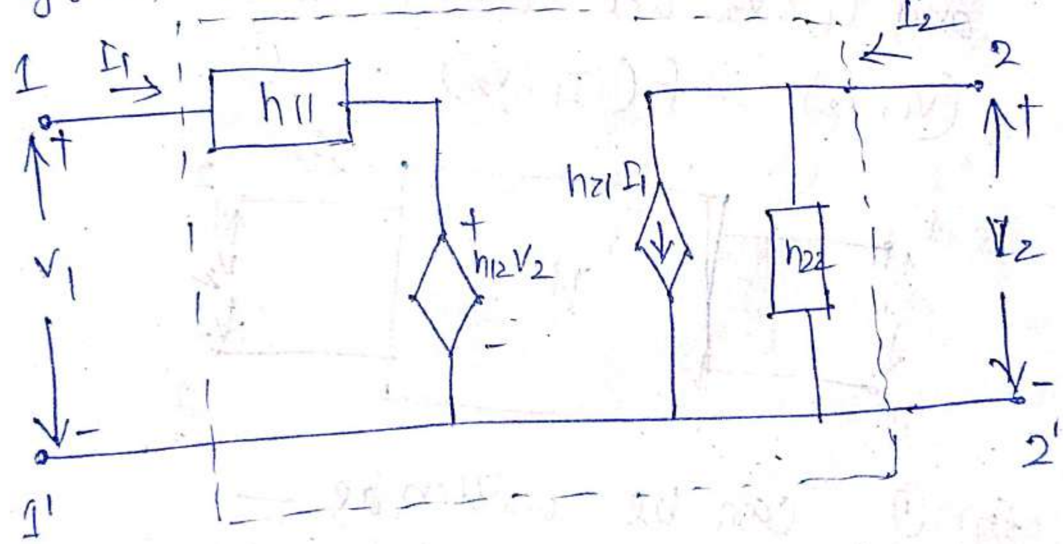
$h_{21} = \frac{I_2}{I_1}\Big|_{V_2=0}$, h_{21} is the forward current gain (h_f), when port II is short ckted.

$h_{22} = \frac{I_2}{V_2}\Big|_{I_1=0}$, h_{22} is the output admittance (h_o) when port I is open ckted.

so eqns ② can be written as —

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_i & h_r \\ h_f & h_o \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

The h-parameter equivalent circuit can be given as —



parameter conversion

a) Z to Y parameters.

Z parameter eqns are —

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad \text{——} ①$$

Y parameter eqns are —

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad \text{——} ②$$

from ① $[V] = [Z][I]$

or $[I] = [Z]^{-1}[V]$ —③

compairing ③ with ② which is $[I] = [Y][V]$.

it is evident that, $[Y] = [Z]^{-1} = \dfrac{Adj[Z]}{|Z|}$ ✓ $= \dfrac{[\text{Cofactor matrix}]^{T}}{|Z|}$

$$= \frac{\begin{bmatrix} Z_{22} & -Z_{12} \\ -Z_{21} & Z_{11} \end{bmatrix}}{Z_{11}Z_{22} - Z_{12}Z_{21}}$$

$$\therefore [Y] = \frac{\begin{bmatrix} Z_{22} & -Z_{12} \\ -Z_{21} & Z_{11} \end{bmatrix}}{\Delta Z} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$$

or $\boxed{y_{11} = \dfrac{Z_{22}}{\Delta Z}}$, $\boxed{y_{12} = -\dfrac{Z_{12}}{\Delta Z}}$ $\boxed{= \dfrac{-Z_{12}}{\Delta Z}}$

$\boxed{y_{21} = -\dfrac{Z_{21}}{\Delta Z}}$, $\boxed{y_{22} = \dfrac{Z_{11}}{\Delta Z}}$

b) Z to h parameters

Z parameter eqns are –

$$V_1 = Z_{11}I_1 + Z_{12}I_2 \quad \text{—①}$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2 \quad \text{—②}$$

h- parameter eqns are –

$$V_1 = h_{11}I_1 + h_{12}V_2 \quad \text{—③}$$
$$I_2 = h_{21}I_1 + h_{22}V_2 \quad \text{—④}$$

from eqn ② $Z_{22}I_2 = -Z_{21}I_1 + V_2$

or $I_2 = -\dfrac{Z_{21}}{Z_{22}}I_1 + \dfrac{1}{Z_{22}}V_2$ —⑤

compairing ⑤ with ④

$\boxed{h_{21} = -\dfrac{Z_{21}}{Z_{22}}}$ and $\boxed{h_{22} = \dfrac{1}{Z_{22}}}$

putting 5 in 1 —

$$V_1 = Z_{11} I_1 + Z_{12}\left[\frac{-Z_{21}}{Z_{22}} I_1 + \frac{1}{Z_{22}} V_2\right]$$

$$= \frac{Z_{22}Z_{11} - Z_{12}Z_{21}}{Z_{22}} I_1 + \frac{Z_{12}}{Z_{22}} V_2 \quad \text{—⑥}$$

compairing 6 & 3

$$\boxed{h_{11} = \frac{Z_{22}Z_{11} - Z_{12}Z_{21}}{Z_{22}}} \text{ and } \boxed{h_{12} = \frac{Z_{12}}{Z_{22}}}$$

c) <u>Y to Z parameters</u>

y parameter eqns are $[I] = [Y][V]$ —①

Z parameter eqns are $[V] = [Z][I]$ —②

from eqn ① $[V] = [Y]^{-1} \cdot [I]$ —③

compairing ③ with ② —

$$[Z] = [Y]^{-1} = \frac{Adj\,[Y]}{|Y|} = \frac{\begin{bmatrix} y_{22} & -y_{12} \\ -y_{21} & y_{11} \end{bmatrix}}{\Delta Y}$$

or $\boxed{Z_{11} = \frac{y_{22}}{\Delta Y}}$ $\boxed{Z_{12} = \frac{-y_{12}}{\Delta Y}}$

$\boxed{Z_{21} = \frac{-y_{21}}{\Delta Y}}$ $\boxed{Z_{22} = \frac{y_{11}}{\Delta Y}}$

(d) <u>y to h-parameters</u>

y parameter eqns are —

$I_1 = y_{11} V_1 + y_{12} V_2$ —①

$I_2 = y_{21} V_1 + y_{22} V_2$ —②

h-parameter eqns are —

$V_1 = h_{11} I_1 + h_{12} V_2$ —③

$I_2 = h_{21} I_1 + h_{22} V_2$ —④

from (1) $y_{11}V_1 = I_1 - y_{12}V_2$

or $V_1 = \frac{1}{y_{11}}I_1 - \frac{y_{12}}{y_{11}}V_2$ —— (5)

compairing (5) with (3).

$\boxed{h_{11} = \frac{1}{y_{11}}}$ and $\boxed{h_{12} = -\frac{y_{12}}{y_{11}}}$

putting (5) in (2) —

$I_2 = y_{21}\left[\frac{1}{y_{11}}I_1 - \frac{y_{12}}{y_{11}}V_2\right] + y_{22}V_2$

$= \frac{y_{21}}{y_{11}}I_1 + \frac{y_{22}y_{11} - y_{12}y_{21}}{y_{11}}V_2$ —— (6)

compairing (6) with (4) —

$\boxed{h_{21} = \frac{y_{21}}{y_{11}}}$ and $\boxed{h_{22} = \frac{y_{22}y_{11} - y_{12}y_{21}}{y_{11}}}$

e) h to Z parameter

h-parameter eqns are —
$V_1 = h_{11}I_1 + h_{12}V_2$ —— (1)
$I_2 = h_{21}I_1 + h_{22}V_2$ —— (2)

Z- parameter eqns are ·
$V_1 = Z_{11}I_1 + Z_{12}I_2$ —— (3)
$V_2 = Z_{21}I_1 + Z_{22}I_2$ —— (4)

from (2) $h_{22}V_2 = I_2 - h_{21}I_1$

or, $V_2 = \frac{1}{h_{22}}I_2 - \frac{h_{21}}{h_{22}}I_1$ —— (5)

compairing (5) & (4).

$\boxed{Z_{21} = -\frac{h_{21}}{h_{22}}}$ and $\boxed{Z_{22} = \frac{1}{h_{22}}}$

putting (5) in (1), $V_1 = h_{11}I_1 + h_{12}\left(\frac{1}{h_{22}}I_2 - \frac{h_{21}}{h_{22}}I_1\right)$

$= \frac{h_{22}h_{11} - h_{12}h_{21}}{h_{22}}I_1 + \frac{h_{12}}{h_{22}}I_2$ —— (6)

compairing (6) & (3)

$\boxed{Z_{11} = \frac{h_{22}h_{11} - h_{12}h_{21}}{h_{22}}}$ and $\boxed{Z_{12} = \frac{h_{12}}{h_{22}}}$

(12) f) h to y parameter:

h-parameter eqns are - $V_1 = h_{11} I_1 + h_{12} V_2$ — (1)

$I_2 = h_{21} I_1 + h_{22} V_2$ — (2)

y-parameter eqns are - $I_1 = y_{11} V_1 + y_{12} V_2$ — (3)

$I_2 = y_{21} V_1 + y_{22} V_2$ — (4)

from (1) $h_{11} I_1 = V_1 - h_{12} V_2$

or, $I_1 = \frac{1}{h_{11}} V_1 - \frac{h_{12}}{h_{11}} V_2$ — (5)

compairing (5) & (3) —

$\boxed{y_{11} = \frac{1}{h_{11}}}$, $\boxed{y_{12} = -\frac{h_{12}}{h_{11}}}$

putting (5) in (2) —

$I_2 = h_{21} \left(\frac{1}{h_{11}} V_1 - \frac{h_{12}}{h_{11}} V_2 \right) + h_{22} V_2$

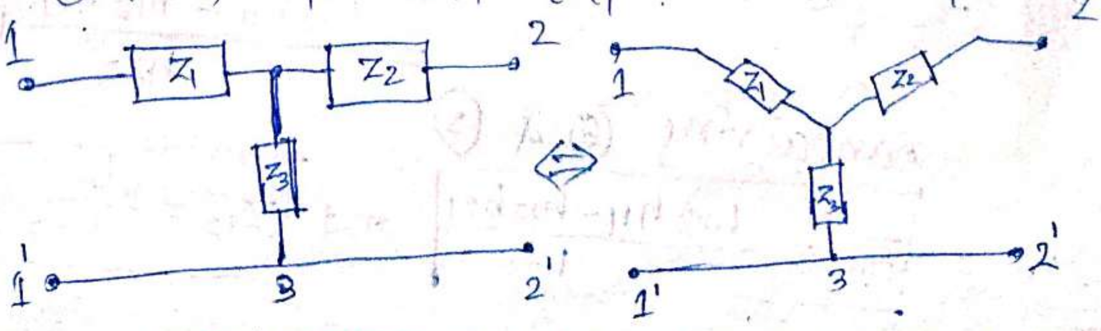
$= \frac{h_{21}}{h_{11}} V_1 + \frac{h_{22} h_{11} - h_{12} h_{21}}{h_{11}} V_2$ — (6)

compairing (4) & (6).

$\boxed{y_{21} = \frac{h_{21}}{h_{11}}}$ and $\boxed{y_{22} = \frac{h_{22} h_{11} - h_{12} h_{21}}{h_{11}}}$

6.4 T and Π (Star & Delta) networks

'T' network is a two port network and it looks like 'T' (capital letter T). Actualy it is a 3 winged star like network and one of the wing is made to two, common to both input and output.

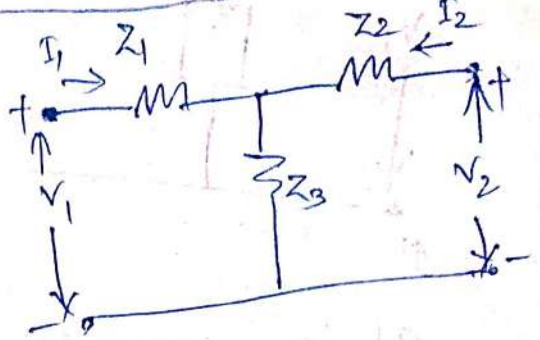


It is easy to find z-parameters of a 'T' network. ~~For~~ any other parameters can be converted from z parameters. ~~z to~~

Z-parameters of T-network

$V_1 = Z_{11} I_1 + Z_{12} I_2$

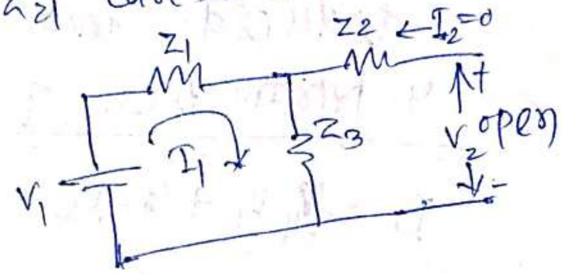
$V_2 = Z_{21} I_1 + Z_{22} I_2$



$Z_{11} = \frac{V_1}{I_1}\Big|_{I_2=0}$, $Z_{12} = \frac{V_1}{I_2}\Big|_{I_1=0}$

$Z_{21} = \frac{V_2}{I_1}\Big|_{I_2=0}$, $Z_{22} = \frac{V_2}{I_2}\Big|_{I_1=0}$.

making $I_2 = 0$, Z_{11} and Z_{21} can be obtained.



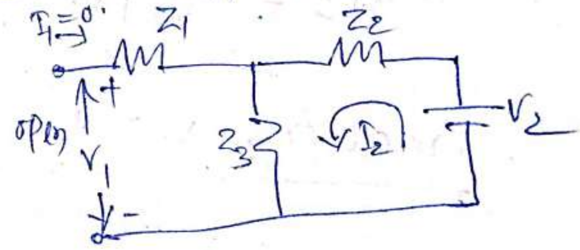
now $V_1 = Z_1 I_1 + Z_3 I_1$

$= (Z_1 + Z_3) I_1$

or $\frac{V_1}{I_1} = Z_1 + Z_3 = Z_{11}$

and $V_2 = I_1 Z_3$ or $\frac{V_2}{I_1} = Z_3 = Z_{21}$.

Similarly making $I_1 = 0$, Z_{12} & Z_{22} can be obtained



now $V_2 = I_2 (Z_2 + Z_3)$

or $\frac{V_2}{I_2} = Z_2 + Z_3 = Z_{22}$

and $V_1 = Z_3 I_2$

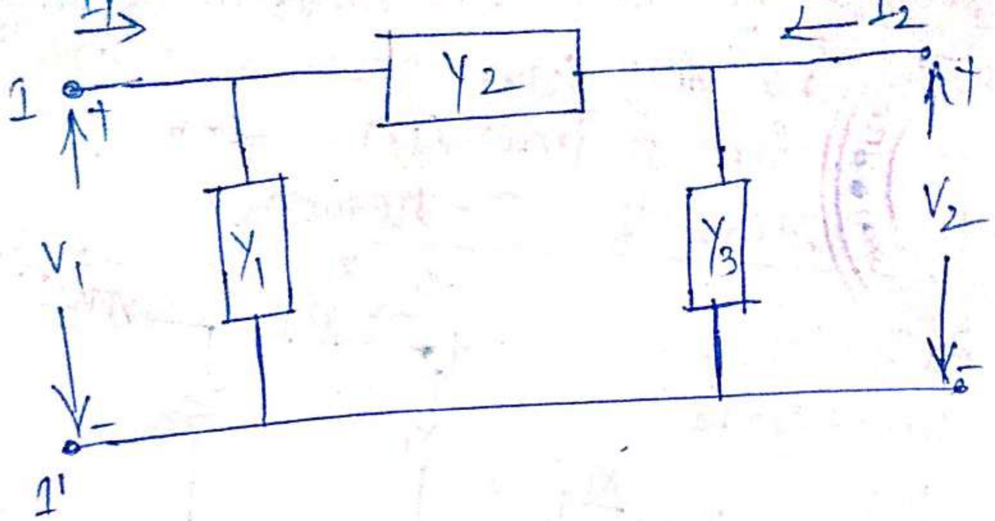
or $\frac{V_1}{I_2} = Z_3 = Z_{12}$.

so, $\boxed{Z_{11} = Z_1 + Z_3}$ $\boxed{Z_{12} = Z_{21} = Z_3}$ $\boxed{Z_{22} = Z_2 + Z_3}$

π (Pie) network

It is a two port network in the from of π (Pie) or delta.

(15).



It is easy to find out y-parameters of a π network. Other parameters can be deduced from y-parameters.

y-parameters of a π network.

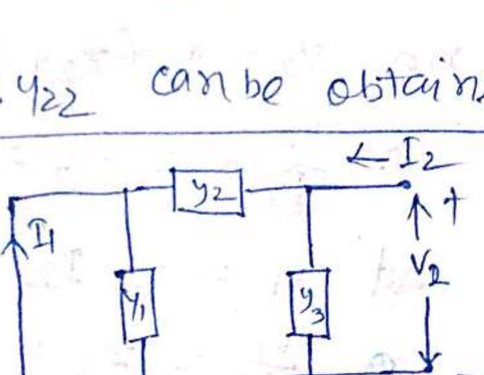
$$I_1 = y_{11} V_1 + y_{12} V_2$$

$$I_2 = y_{21} V_1 + y_{22} V_2$$

$$y_{11} = \frac{I_1}{V_1}\Big|_{V_2=0}, \quad y_{21} = \frac{I_2}{V_1}\Big|_{V_2=0}.$$

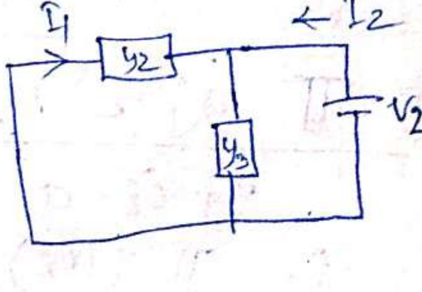
$$y_{12} = \frac{I_1}{V_2}\Big|_{V_1=0}, \quad y_{22} = \frac{I_2}{V_2}\Big|_{V_1=0}.$$

making $V_1 = 0$, y_{12} & y_{22} can be obtained



now $I_2 = (y_2 + y_3) V_2$

or $\dfrac{I_2}{V_2} = y_2 + y_3 = y_{22}$



and $I_1 = -I_2 \times \dfrac{1/y_3}{\frac{1}{y_2} + 1/y_3}$

$$= -I_2 \cdot \frac{1/y_3}{\frac{y_2 + y_3}{y_2 y_3}}$$

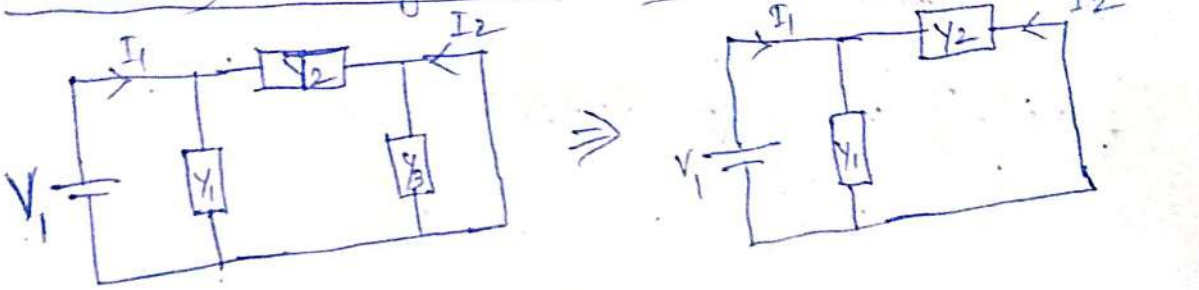
$$= -I_2 \cdot \frac{1}{y_3} \times \frac{y_2 y_3}{y_2+y_3} = -I_2 \cdot \frac{y_2}{y_2+y_3}$$

putting $I_2 = (y_2+y_3)V_2$ —

$$I_1 = -(y_2+y_3)V_2 \cdot \frac{y_2}{(y_2+y_3)} = -V_2 y_2$$

$$\therefore \frac{I_1}{V_2} = -y_2 = y_{12}.$$

Similarly making $V_2 = 0$, y_{21} and y_{11} can be obtained.



now $I_1 = V_1(y_1+y_2)$

or $\frac{I_1}{V_1} = y_1+y_2 = y_{11}$.

and $I_2 = -I_1 \times \frac{Y y_1}{\frac{1}{y_1}+\frac{1}{y_2}} = -I_1 \cdot \frac{Y y_1}{\frac{y_1+y_2}{y_1 \cdot y_2}}$

$$= -I_1 \frac{y_2}{y_1+y_2}$$

putting $I_1 = (y_1+y_2)V_1$ —

$$I_2 = -(y_1+y_2)V_1 \cdot \frac{y_2}{(y_1+y_2)} = -y_2 V_1$$

or $\frac{I_2}{V_1} = -y_2 = y_{21}$

$\therefore$ $\boxed{y_{11} = y_1+y_2}$ $\boxed{y_{12} = y_{21} = -y_2}$ $\boxed{y_{22} = y_2+y_3}$.

$$A^{-1} = \frac{Adj\,[A]}{|A|}$$

$$Adj[A] = [cofactor]^T$$

$$[A] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

cofactor of $A_{11} = A_{22}$

" $A_{12} = -A_{21}$

" $A_{21} = -A_{12}$

" $A_{22} = A_{11}$

cofactor matrix $= \begin{bmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{bmatrix}$

Transpose of cofactor matrix $A = \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$

$= Adj[A]$

$mod\,A = |A| = A_{11} \cdot A_{22} - A_{12} A_{21}$

Unit 7

Filters And Attenuators. ①

7.1 Ideal and practical filters, its application:

A filter is a frequency selecting electrical network (or device) which allows a band (range) of frequency components to pass through it (or to be stopped by it) while stoping all other frequency components (or passing all other frequency compnts)

Pass band: The band of frequencies which pass through the filter is called pass band.

stop band or attenuation band: The band of frequencies which is suppressed is called stop band or attenuation band.

cut off frequency: The frequency that separates passband and stop band is called cut off frequency (f_c)

- There may be only one cut off frequency (f_c). If the passband is below this frequency (then stop band is above it), then it is called upper cut off frequency (f_H) and if the passband is above this frequency (so stop band is below), then it is called lower cutt off frequency (f_L).
- Such filters may be called as low pass filter LPF (having upper or higher cut off frequency f_H) and high pass filter HPF (having lower cut of freq. f_L)
- Some filters (Band pass filter BPF and band stop filter BSF) have two cut off frequencies f_1 and f_2 (called as lower and upper cut off frequencies)

(2) Ideal filter: An ideal filter is one which passes or transmits signals in the passband without attenuation and completely suppresses (stops) signals in the stop band with a sharp cut off frequency (profile)

Practical Filter: An ideal filter is not practically possible. Hence a practical filter is one which do not completely pass the signal in the pass band (some signal get attenuated due to absorption, reflection or other loss) and do not completely stops the signal in attenuation band (some portion may get passed). Hence such filter dont have a sharp cut off profile or frequency.

Properties of filter: followings are the properties of passive filters.

a) characteristic impedance: (Z_0) characteristic impedance of a filter is the impedance which matches to the circuit to which it is connected throughout the passband.

b) pass band (and stop band) characteristics: The filter should have minimum (or zero) attenuation in the passband and high attenuation in the stop band range.

c) cut off frequency characteristics: The filter should have strict frequency selective property. It should be capable of identifying lower as well as higher cutoff frequency for transmitting signal through it.

③ Application of filters:

1. In voice frequency telegraphy, multichannel communication is possible by utilising a number of band pass filters (BPFs) with different pass bands.
2. In Telephony or in radio and TV broadcasting several number of information are transmitted by modulating different carrier frequencies which can be received by utilising filter circuits.
3. In radio receivers intermediate carrier frequency selection is made by using filters.
4. In AM detection. HPFs are used to separate carrier frequency signal from audio frequency.
5. In TV receivers, filters are used to select horizontal and vertical sync. pulses from composite video signal.
6. In Audio amplifiers, filters are used to reduce harmonic distortion and voice rejection.
7. In regulated power supply, filters are used to provide smooth dc output from ac signals.

7.2 Classification of filters:

Filters can be classified in two ways—

a) depending upon the relation between series arm impedance Z_1 and shunt arm impedance Z_2 filters are catagorised as—

i) constant K-filter or proto-type filter.

ii) m-derived filter.

b) depending upon their frequency characteristics The filters are classified as—

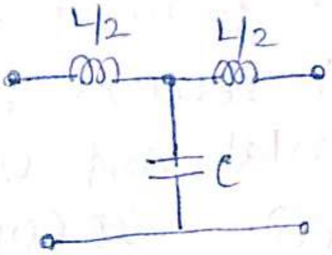
(4) i) Low pass filter (LPF) ii) High pass filter (HPF)
iii) Band pass filter (BPF)
iv) Band stop (elimination/rejection/attenuation) filter (BSF)

Constant K filter (prototype filter):

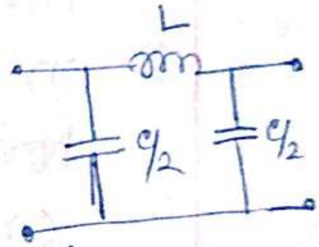
In prototype filters, the series arm & shunt arm impedances are such that $Z_1 \cdot Z_2 = R_0^2 = K$
where $R_0 \rightarrow$ is called the design impedance and is a real quantity independent of frequency.

It can be further classified to LPF, HPF, BPF and BSF.

i) Constant K- LPF:

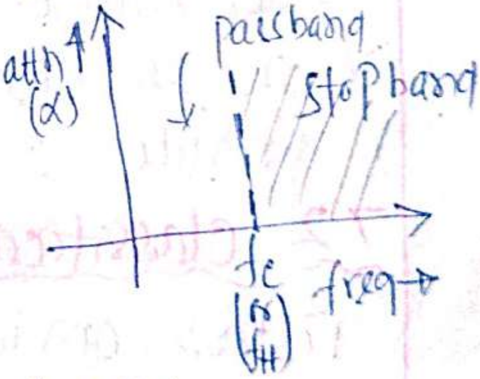


(T- configuration of LPF)



(π- configuration of LPF)

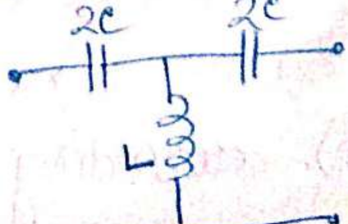
In the figures are shown the T-configuration and π- configuration of constant K- Low pass filters.



The attenuation characteristic of the LPF is as shown in the figure. The cut off freq is the upper (higher) cut off freq (f_H), below which all the frequencies pass and above which all the frequency components get attenuated (stopped).

ii) Constant K-HPF:

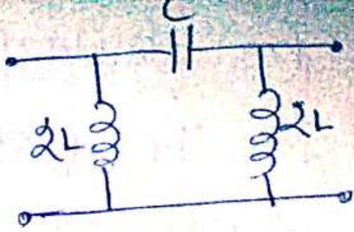
In the figures are shown T and π- configurations of HPF.



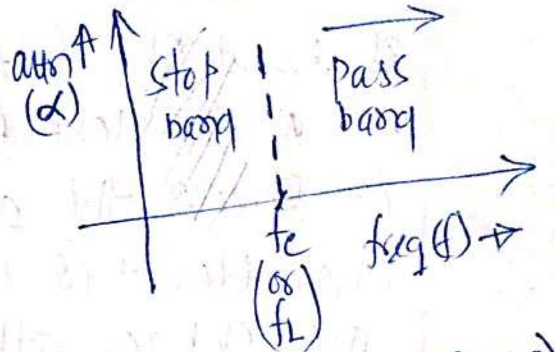
(T- configuration HPF)

(5) In the last figure is shown the attenuation characteristics.

The cut off frequency also called lower cut off frequency (f_L) because all the frequency components below f_L are stopped (or attenuated) while frequencies above f_L passed through the filter.



(π configuration of HPF)

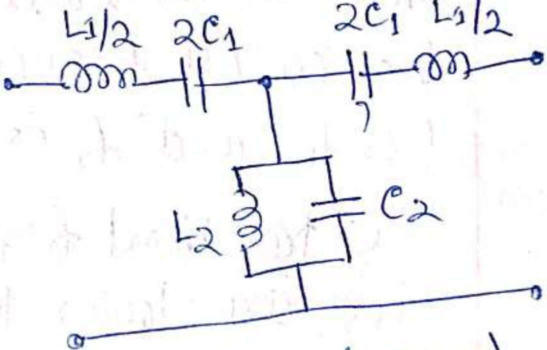


(attenuation ch. of HPF)

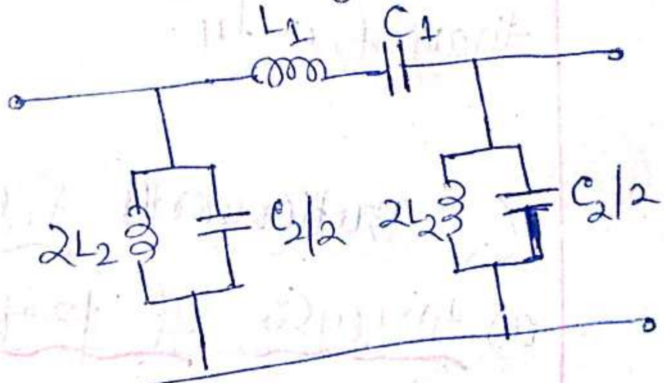
iii) Constant-K BPF.

In first two figures are shown the T & π configurations of Band pass filter (BPFs) respectively.

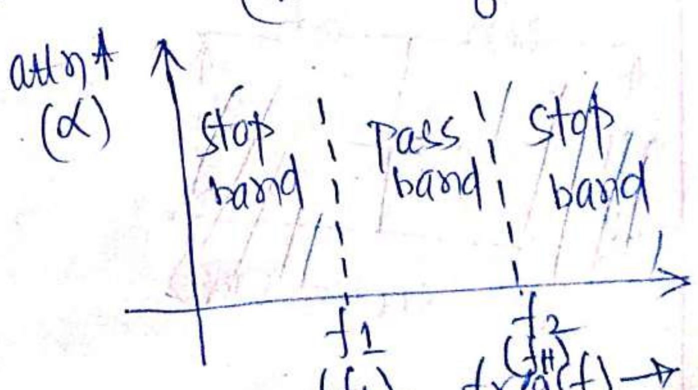
The pass band is between the two cutoff frequencies, f_1 or f_L and f_2 or f_H. Stop band frequencies are below f_1 and above f_2. The attenuation characteristics is shown in last figure.



(T config. of BPF)



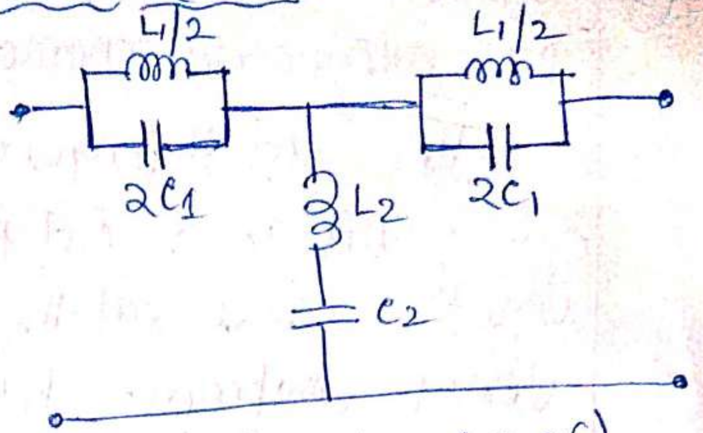
(π- config. of BPF)



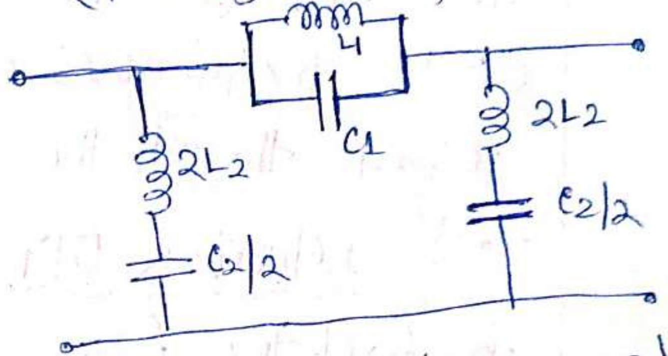
(attn. ch. of BPF)

⑥ (iv) Constant K Band Stop filter (BSF):

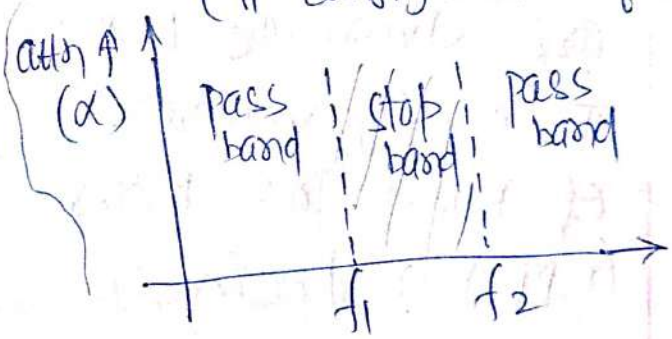
In the first two figures are shown the T and π configurations of Band Stop filter (BSF). In the last figure is shown the attenuation characteristics of BSF. The stop (or attenuation) band is in between the two cutoff frequencies. f_1 or f_L and f_2 or f_H. The passband frequencies are lower than f_1 or f_L and higher than f_2 or f_H.



(T-configuration of BSF)



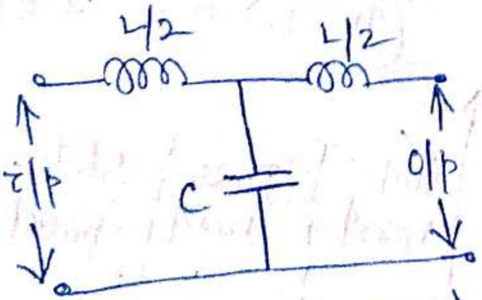
(π-configuration of BSF)



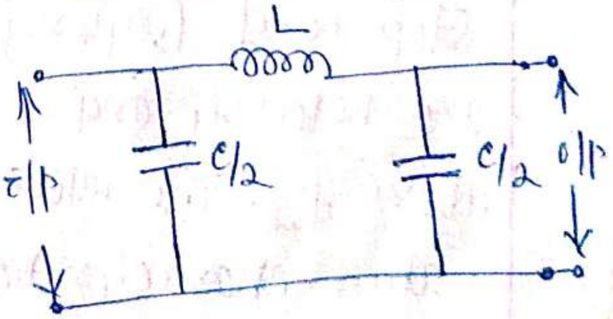
7.3. Butterworth Filter Design:

a) Analysis of prototype Low Pass filter (LPF)

T and π sections of a LPF can be as follows –



(T- section LPF)



(π- section LPF)

In T or π section —

total series impedance $Z_1 = j\omega L$ —— ①

⑦ and total shunt impedance $Z_2 = \frac{1}{j\omega C} = -j/\omega C$ —②

multiplying both eqns ① & ②,

$$Z_1 \cdot Z_2 = j\omega L\left(\frac{-j}{\omega C}\right) = \frac{L}{C} = R_0^2 \quad \text{—③}$$

(where R_0 is a constant called design impedance)

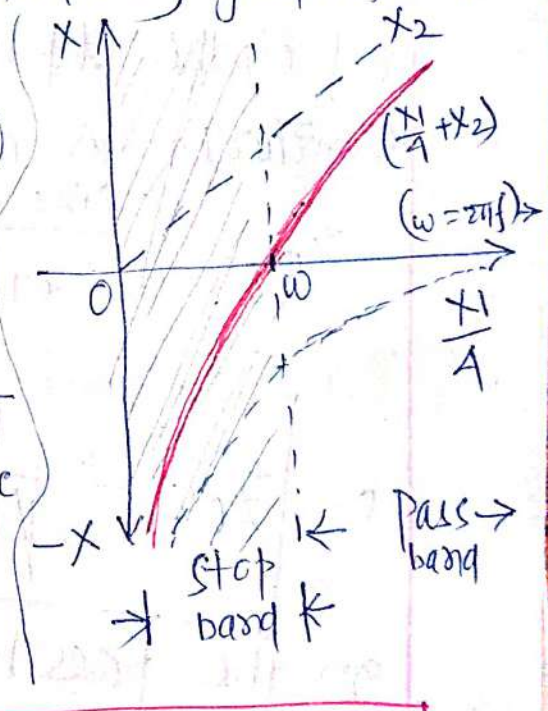
now $$\frac{Z_1}{4Z_2} = \frac{j\omega L}{-4j/\omega C} = \frac{-\omega^2 LC}{4} \quad \text{—④}$$

for a two port network, the characteristic impedance for a T- section is $Z_{0T} = \sqrt{Z_1 Z_2\left(1+\frac{Z_1}{4Z_2}\right)}$ —⑤

where Z_0 is called the characteristic impedance.

The reactance versus (X V_s ω) frequency graph for a LP filter can be as shown.

putting the values from ③ & ④ in ⑤, we get —



$$Z_{0T} = \sqrt{L/C}\sqrt{1-\frac{\omega^2 LC}{4}}$$

$$= R_0\sqrt{1-\frac{\omega^2 LC}{4}} = R_0\sqrt{1-\frac{\omega^2}{4/LC}}$$

putting $\omega_c^2 = \frac{4}{LC}$ or $\omega_c = \frac{2}{\sqrt{LC}}$

$\omega_c \to$ cutoff frequency.

$$\boxed{Z_{0T} = R_0\sqrt{1-\frac{\omega^2}{\omega_c^2}}} \quad \text{or,} \quad \boxed{Z_{0T} = R_0\sqrt{1-(f/f_c)^2}}$$

From the above it is evident that, $Z_{0T} = R_0\sqrt{1-\frac{\omega^2 LC}{4}}$ is real, if $\frac{\omega^2 LC}{4} < 1$ and imaginary if $\frac{\omega^2 LC}{4} > 1$ i.e, Z_{0T} indicates the characteristic impedance of the pass band when $\frac{\omega^2 LC}{4} < 1$ (as Z_{0T} remains real) and Z_{0T} becomes the characteristic impedance of

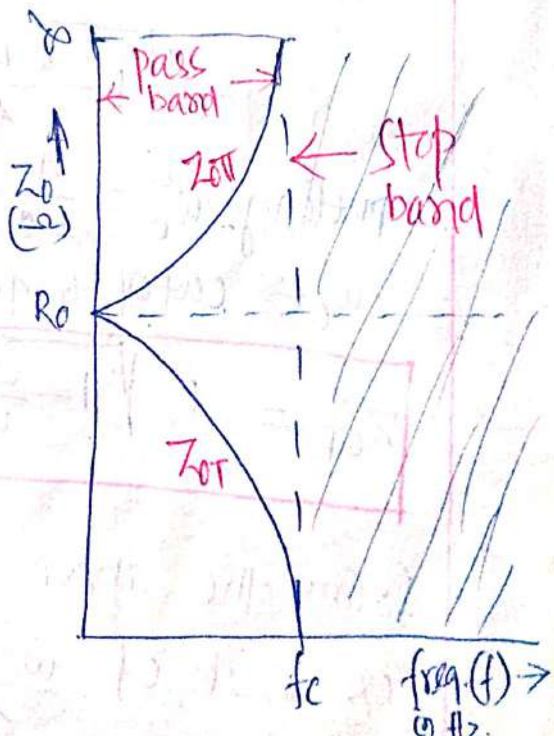
(B) the stop band if $\frac{\omega^2 LC}{4} > 1$ (as Z_{0T} becomes imaginary)
Hence it is obvious that cut off frequency will be at a particular condition when $\frac{\omega^2 LC}{4} = 1$
that is $\omega_c = \frac{2}{\sqrt{LC}}$.

Hence cutoff frequency of a LPF is $\boxed{\omega_c = \frac{2}{\sqrt{LC}}}$
or, $\boxed{f_c = \frac{1}{\pi\sqrt{LC}}}$

It is also evident that, ~~in the~~ —
in the pass band — when Z_{0T} is real, $f < f_c$
and in the stop band — when Z_{0T} is imaginary, $f > f_c$
Similarly for a π-section (from two-port network)

$$Z_{0\pi} = \frac{Z_1 Z_2}{Z_{0T}} = \frac{R_0^2}{R_0\sqrt{1-(f/f_c)^2}} = \frac{R_0}{\sqrt{1-(f/f_c)^2}}$$

or $\boxed{Z_{0\pi} = \frac{R_0}{\sqrt{1-\left(\frac{f}{f_c}\right)^2}}}$



In the pass band, $Z_{0\pi}$ is real and +ve quantity, and in the stop band, $Z_{0\pi}$ is imaginary.
In the fig. is shown the characteristic impedance profile of T and π section of an LPF.

(9) Attenuation (α) and phase shift (β) ch. of LPF.

From two port network theory, $\cosh\gamma = 1+\frac{Z_1}{2Z_2}$

where $\gamma = \alpha + j\beta$

In a filter section, $Z_1 = jX_1$ and $Z_2 = jX_2$ where X_1 and X_2 are the reactances of series and shunt arm. for an LPF, $X_1 = j\omega L$ and $X_2 = -j/\omega C$

for a low pass filter T section, $1+\frac{Z_1}{2Z_2} = 1-\frac{\omega^2 LC}{2}$

$\left[\because \frac{Z_1}{4Z_2} = -\frac{\omega^2 LC}{4}\right]$

now $\cosh\gamma = 1-\frac{\omega^2 LC}{2}$

or, $\cosh(\alpha+j\beta) = 1-\frac{\omega^2 LC}{2}$

or, $\cosh\alpha\cdot\cos\beta + j\sinh\alpha\cdot\sin\beta = 1-\frac{\omega^2 LC}{2}$ —(1)

Equating real and imaginary parts of eqn (1),

$\cosh\alpha\cdot\cos\beta = 1-\frac{\omega^2 LC}{2}$ —(2) and $\sinh\alpha\cdot\sin\beta = 0$ —(3)

From eqn (3), either $\alpha=0$ or $\beta = n\pi$ (n being an integer)

(i) For pass band:

In the pass band, there is no attenuation i.e, $\alpha=0$,

Then from eqn (2) above. $\cos\beta = 1-\frac{\omega^2 LC}{2}$ —(4)

The value of $\cos\beta$ swings from -1 to $+1$.

⇒ putting $\cos\beta = 1$, in eqn (4) above gives -

$\frac{\omega^2 LC}{2} = 0$ or $\omega = 0$ or $f=0$, which is the lower cut off frequency of the filter operation,

⇒ Putting $\cos\beta = -1$, in eqn (4), $-1 = 1-\frac{\omega^2 LC}{2}$

or, $\frac{\omega^2 LC}{2} = 2$ or $\omega^2 LC = 4$ or $\omega^2 = \frac{4}{LC}$ and $\boxed{\omega = \frac{2}{\sqrt{LC}}}$

(10) Thus the higher cut off frequency of the low pass filter is given by $\boxed{f_c = \frac{1}{\pi\sqrt{LC}}}$ or, $\boxed{\omega_c = \frac{2}{\sqrt{LC}}}$

Again from eqn (1) $\cos\beta = 1 - \frac{\omega^2 LC}{2} = 1 - \frac{\omega^2}{2/LC} = 1 - 2\frac{\omega^2}{\omega_c^2}$

$\therefore \boxed{\beta = \cos^{-1}\left[1 - 2\left(\frac{\omega^2}{\omega_c^2}\right)\right]}$ radian —— (5)

Also $\cos\beta = 1 - 2\sin^2\frac{\beta}{2} = 1 - 2\frac{\omega^2}{\omega_c^2}$

$\therefore \sin^2\frac{\beta}{2} = \frac{\omega^2}{\omega_c^2}$ or $\sin\frac{\beta}{2} = \frac{\omega}{\omega_c}$ or $\boxed{\beta = 2\sin^{-1}\left(\frac{\omega}{\omega_c}\right)}$ radian

(ii) For attenuation Band

In attenuation band $\alpha \neq 0$, so from eqn (3) it is obvious that $\beta = n\pi$.

for $n = 1$, $\beta = \pi$, putting in eqn (2), $\cos n\beta = \cos\pi = -1$

$-\cosh\alpha = 1 - 2\frac{\omega^2}{\omega_c^2}$ or, $\boxed{\alpha = \cosh^{-1}\left[2\left(\frac{\omega^2}{\omega_c^2}\right) - 1\right]}$ neper —(6)

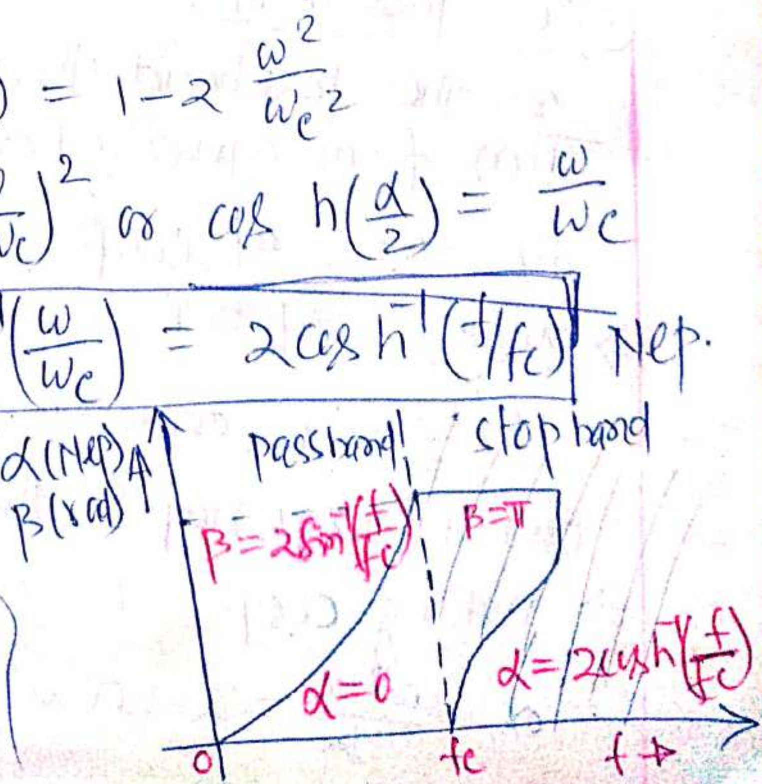
Also $-\cosh\alpha = 1 - 2\cosh^2(\alpha/2)$

or, $1 - 2\cosh^2\left(\frac{\alpha}{2}\right) = 1 - 2\frac{\omega^2}{\omega_c^2}$

or, $\cosh^2\left(\frac{\alpha}{2}\right) = \left(\frac{\omega}{\omega_c}\right)^2$ or $\cosh\left(\frac{\alpha}{2}\right) = \frac{\omega}{\omega_c}$

or $\boxed{\alpha = 2\cosh^{-1}\left(\frac{\omega}{\omega_c}\right) = 2\cosh^{-1}(f/f_c)}$ Nep.

The attenuation (α) and phase shift (β) characteristics of a LPF is as shown in figure.



(ii) Design of prototype (constant K) LPF.

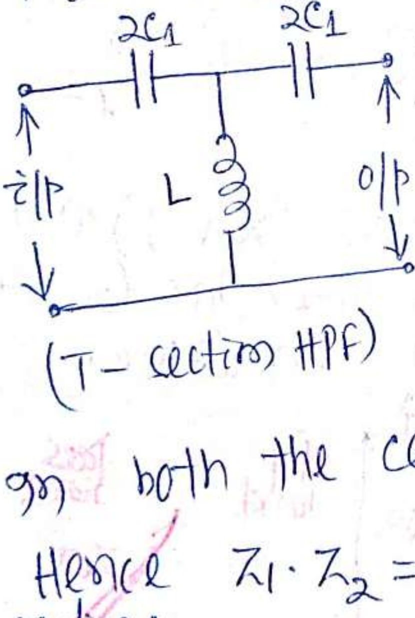
To design a prototype LPF, for a particular value of design (resistance) impedance R_0 and cut off frequency f_c, the inductance and capacitance values can be obtained from them as-

$$\boxed{R_0 = \sqrt{L/C}} \quad \text{and} \quad \boxed{f_c = \frac{1}{\pi\sqrt{LC}}}$$

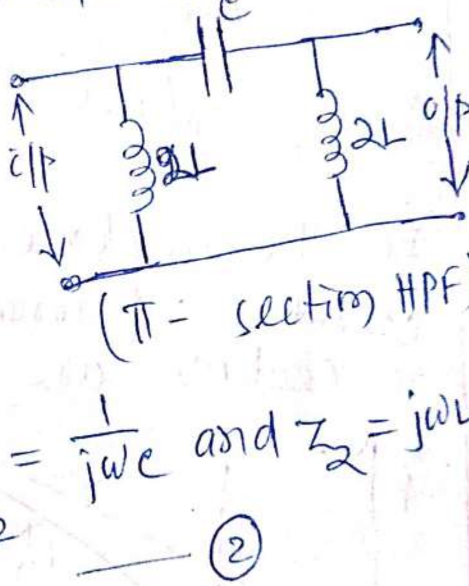
then, $\boxed{L = \frac{R_0}{\pi f_c}}$ and $\boxed{C = \frac{1}{\pi R_0 f_c}}$

(b) Analysis of a prototype high pass filter (HPF)

T and π sections of a prototype HPF are as shown-



(T- section HPF)



(π- section HPF)

In both the cases, $Z_1 = \frac{1}{j\omega C}$ and $Z_2 = j\omega L$ —(1)

Hence $Z_1 \cdot Z_2 = \frac{L}{C} = R_0^2$ —(2)

For T-section: The characteristic impedance of a T-section is given by

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2} \quad —(3)$$

putting the values of Z_1, Z_2 and Z_1Z_2 from (1) & (2) in (3)

$$Z_{0T} = \sqrt{-\frac{1}{4\omega^2C^2} + \frac{L}{C}} = \sqrt{\frac{L}{C}}\sqrt{1 - \frac{1}{4\omega^2 CL}}$$

$$= R_0\sqrt{1 - \frac{1}{4\omega^2 LC}} \quad —(4)$$

(12)

⇒ If, $4\omega^2 LC > 1$, Z_{0T} is real and the filter works in the pass band.

⇒ If $4\omega^2 LC < 1$, Z_{0T} is imaginary and the filter works in the stop band.

⇒ and thus $4\omega^2 LC = 1$, gives the cut off frequency as

$$\boxed{\omega_c = \frac{1}{2\sqrt{LC}}} \quad \text{or,} \quad \boxed{f_c = \frac{1}{4\pi\sqrt{LC}}} \qquad \text{— (5)}$$

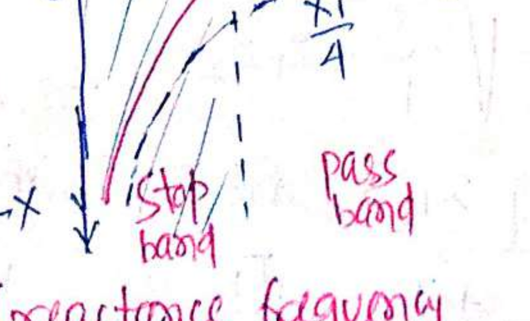
putting (5) in (4), $Z_{0T} = R_0\sqrt{1 - \frac{1}{4\omega^2 LC}}$

$$= R_0\sqrt{1 - \frac{\omega_c^2}{\omega^2}} = R_0\sqrt{1 - \frac{f_c^2}{f^2}} \qquad \text{— (6)}$$

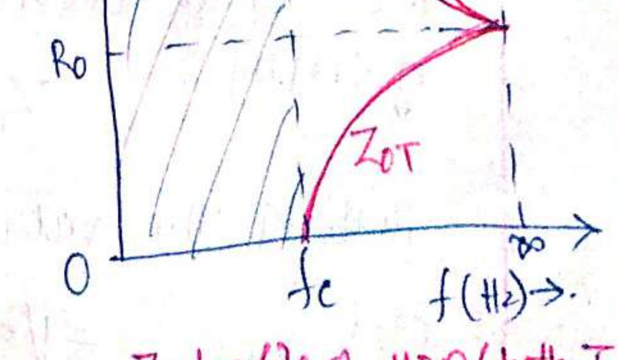
<u>for π- section:</u> $Z_{0\pi} = \frac{R_0^2}{Z_{0T}}$

$$= \frac{R_0^2}{R_0\sqrt{1 - \frac{\omega_c^2}{\omega^2}}} = \frac{R_0}{\sqrt{1 - \frac{\omega_c^2}{\omega^2}}} = \frac{R_0}{\sqrt{1 - \frac{f_c^2}{f^2}}} \qquad \text{— (7)}$$

Thus the reactance characteristics (X v_s f) and characteristic impedance ~~freq~~ characteristics for both T and π sections are as shown in figure.



(reactance frequency characteristics of HPF)



Z_0 profile of HPF (both T and π section)

(13) **Attenuation (α) and phaseshift (β) characteristics of HPF:**

for a T-network the propagation constant γ is given by $\cosh\gamma = 1 + \frac{Z_1}{2Z_2}$ putting $Z_1 = \frac{1}{j\omega C}$, $Z_2 = j\omega L$)

$$= 1 + \frac{1/j\omega C}{2j\omega L} = 1 + \frac{1}{j\omega C} \times \frac{1}{2j\omega L} = 1 - \frac{1}{2\omega^2 LC} \quad \text{—(1)}$$

Expanding $\cosh\gamma = \cosh(\alpha + j\beta) = \cosh\alpha \cdot \cos\beta + j\sinh\alpha \cdot \sin\beta$ —(2).

compairing real and imaginary parts of (1) & (2)

$\sinh\alpha \cdot \sin\beta = 0$; —(3) and $\cosh\alpha \cdot \cos\beta = 1 - \frac{1}{2\omega^2 LC}$ —(4) &

from eqn (3) it is evident that —

either $\alpha = 0$ or $\beta = n\pi$ (n is an integer)

i) For passband : In passband $\alpha = 0$,

putting in eqn (4), $\cos\beta = 1 - \frac{1}{2\omega^2 LC}$ —(5)

As value of $\cos\beta$ swings from -1 to $+1$, but $\cos\beta = 1$ gives the cut off frequency. putting $\cos\beta = 1$ in (5)

we have, $1 = 1 - \frac{1}{2\omega^2 LC}$ or $\frac{1}{2\omega^2 LC} = 0$, or $\omega = \infty$ or $f = \infty$,

which is the higher cut off frequency.

Assuming $\cos\beta = -1$, putting in eqn (5) —

$$-1 = 1 - 1/2\omega^2 LC \Rightarrow \omega_c^2 = \frac{1}{4LC} \text{ or } \omega_c = \frac{1}{2\sqrt{LC}}$$

Hence the lower cutoff frequency is —

$$\boxed{\omega_c = \frac{1}{2\sqrt{LC}}} \quad \text{or,} \quad \boxed{f_c = \frac{1}{4\pi\sqrt{LC}}}$$

Also from $\cos\beta = 1 - \frac{1}{2\omega^2 LC} = 1 - 2\frac{\omega_c^2}{\omega^2}$

$$\text{or } \boxed{\beta = \cos^{-1}\left[1 - 2\left(\frac{\omega_c^2}{\omega^2}\right)\right]} \quad \text{or} \quad \boxed{\beta = \cos^{-1}\left[1 - 2\left(\frac{f_c^2}{f^2}\right)\right]}$$

Also, $\cos\beta = 1 - 2\sin^2\frac{\beta}{2} = 1 - 2\frac{\omega_c^2}{\omega^2}$ or $\sin^2(\beta/2) = \left(\frac{\omega_c}{\omega}\right)^2$

(1A) or, $\sin\left(\frac{\beta}{2}\right) = \frac{\omega_c}{\omega}$ or $\boxed{\beta = 2\sin^{-1}\left(\frac{\omega_c}{\omega}\right)}$ radian.

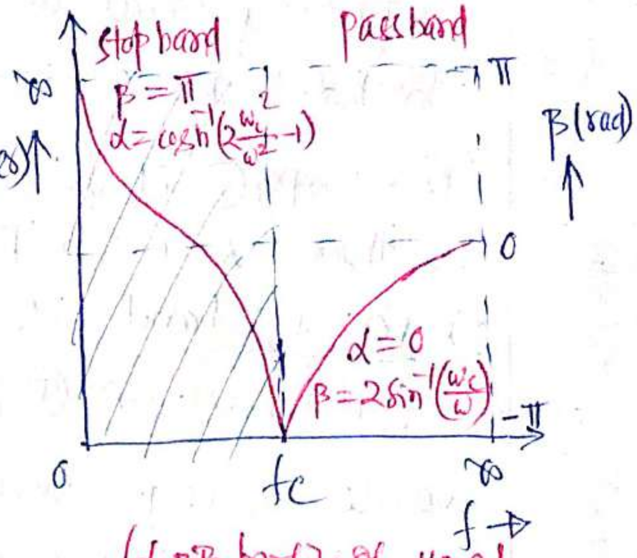
ii) For attenuation band:

putting $n=1$, and so $\beta = \pi$, in eqn (4)

$$\cosh\alpha = -1 + \frac{1}{2\omega^2 LC} = -1 + 2\frac{\omega_c^2}{\omega^2}$$

or $\boxed{\alpha = \cosh^{-1}\left(2\frac{\omega_c^2}{\omega^2} - 1\right)}$ neper when $\beta = \pi$.

The attenuation (α) and phase shif constant (β), profile of HPF is as shown in figure.



(α & β profile of HPF)

Design of prototype HPF.

For a particular value of design impedance R_0 and cut off frequency f_c, the value of L & C can be given as follows - $\boxed{R_0 = \sqrt{L/C}}$ and $\boxed{f_c = \frac{1}{4\pi\sqrt{LC}}}$

$\therefore$ $\boxed{L = \frac{R_0}{4\pi f_c}}$ and $\boxed{C = \frac{1}{4\pi R_0 f_c}}$

(15) ## 7.1 Attenuation and gain : Bel, decibel & neper

A low signal (having less signal strength and so less power) is required to be amplified (raising its strength or power) before transmission over a network.

The factor by which the strength or power of a signal is amplified is called amplification factor or gain of the amplifier.

Gain may be defined as the ratio of laplace (or fourier) transform of output signal to that of the input signal.

$$\therefore \text{Gain} = \frac{L(o/p)}{L(i/p)} = \frac{F(o/p)}{F(i/p)}$$

The reverse of gain is called attenuation or reduction. This occurs when the gain is less than 1.

Attenuation is the loss or reduction in signal strength or power.

Bel, decibel: when analysing a transmission network (circuit) in frequency domain it is required to represent the ratio of two identical signals or parameters of signals (as in case of gain/attenuation). Then this ratio becomes unitless or dimension less.

(12)

Such unitless ratios are represented in a natural logarithemic (to the base 10) base and is called to be in the unit of bel or decibel.

[deci means one tenth, ∴ 1 bel = 10 decibel(dB)

The ~~name~~ term bel (or Bel) has come from the name of the great inventor Alexander Graham Bell).

EX: Let ~~the~~ for a transmission network—
input power = P_1, output power = P_2

Then power gain = $\frac{P_2}{P_1}$

But power gain in decibel or dB = $10 \log_{10}\left[\frac{P_2}{P_1}\right]$

$$\therefore \boxed{dB\ gain = 10 \log_{10} \frac{P_2}{P_1}}$$

if $P_2 = P_1$, o/p = i/p, then $\frac{P_2}{P_1} = 1$.

and dB gain $= 10 \log_{10}(1) = 10 \times 0 = 0$

linear power ratio of 1 corresponds dB gain of 0 dB.

nepers: like Bel or deciBel, neper is also a logarithemic scale used to express ratios (of power, voltage or current) in electrical or electronic networks.

(37) Neper is named after the Scottish mathematician John Napier (1550-1617) whose Latin name was Loannes Neper who has invented logarithems.

For the natural logarithmic base is 'e' instead of 10. where 'e' is called the Euler's number $e = 2.71828\ldots$

So, for a system having input & output powers P_1, P_2 respectively, then gain in

$$\boxed{\text{neper gain} = \log_e (P_2/P_1)}$$

Conversion

$$\boxed{1\ Np = 20 \log_{10} e\ dB}$$

or

$$\boxed{1\ Np = 8.6858896\ dB}$$

$1\ dB = 0.12\ Np$ (ratio $= 1.259$)

and $20\ dB = 2.30\ Np$ (ratio $= 100$)

7.5 Attenuators and its application:

An attenuator is an electronic device that reduces the power of a signal without appreciably distorting its waveform.

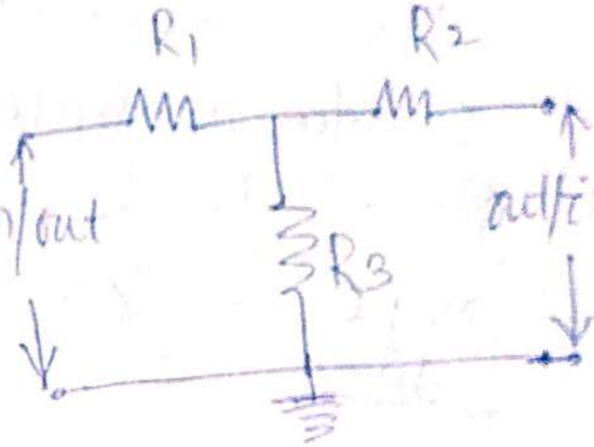
It is the opposite of amplifier, where amplifier produces gain, attenuator

(18) produces loss or reduction of signal power though both of them work on different principles.

Attenuator can be of 2 types –
(1) T- attenuator (2) π- attenuator.

T- attenuator.

In the figure is shown a T- attenuator circuit.



Generally R_1 and R_2 resistances are same so the circuit is symmetrical and used for equal circuits on either sides.

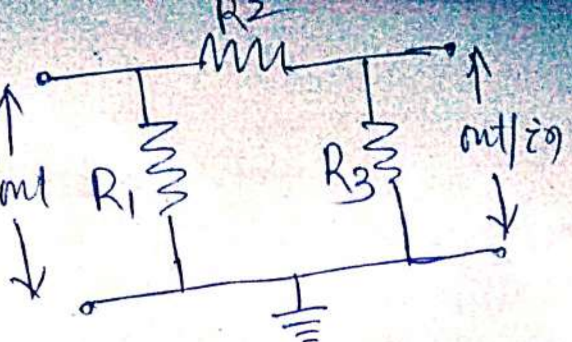
For unequal circuits on both sides, the resistances R_1 and R_2 will be different.

The K- factor or impedance factor of a T- attenuator can be given as –

$$\boxed{K = \text{antilog}\left(\frac{dB}{20}\right) = 10^{\frac{dB}{20}}}$$ for voltage or current

and $$\boxed{K = \text{antilog}\left(\frac{dB}{10}\right) = 10^{\frac{dB}{10}}}$$ for power

(19) ## π attenuator:



π- attenuator is another fully symmetrical purely resistive network. Generally R1 & R2 resistances are equal, but when designed to operate between circuits of unequal impedance these two resistors can have different values.

Attenuator Applications:

- fixed attenuators are used in circuits for — a) lowering voltage level
  b) dissipate power
  c) to improve impedance matching
- In measuring signals, attenuator pads or adapters are used to lower the amplitude of the signal by a known amount to enable measurements.
- Attenuators are also used to protect the measuring device from signals levels, that might damage it.
- Attenuators are also used in RF and optical applications.