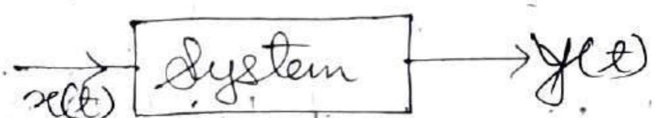
I. Introduction of Signals, Systems and Signal Processing

SIGNAL:- A signal is a physical quantity which varies with respect to some independent variable such as time.

→ Each signal must contain some information.
Example — Speech signal, video signal.

→ Each signal is a function of some independent variable such as time, space, etc

SYSTEM:- A system is an entity which process some input signal and provides some output signal.



$$y(t) = f[x(t)]$$
$$= T[x(t)]$$

→ A system produces o/p inresponse to an I/p signal.
→ A system be can of analog type or digital.
Example:- Analog to digital converter, amplifier.

SIGNAL PROCESSING:-

→ Signal processing is any operation that changes the characteristic of a signal (amplitude, frequency, phase, etc)

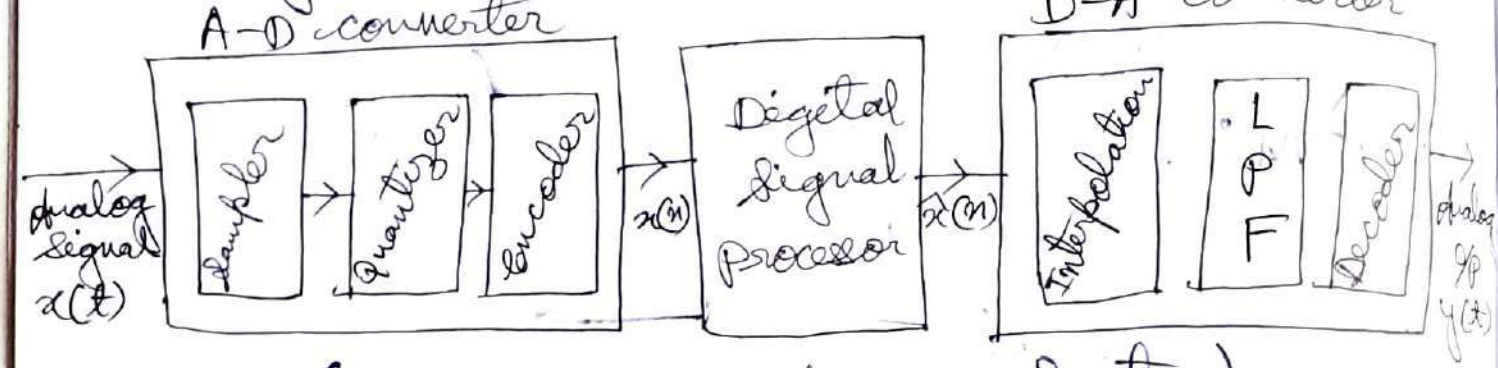
→ Signal processing can be a method of extracting the information from the signal.

→ Signal processing can be of two types:

i) Analog signal processing
ii) Digital signal processing.

Example of analog signal processing — Public address system
Example of digital signal processing — Fingerprint recognition, RADAR.

Basic Elements of a Digital Signal Processing System :-



(Block Diagram of DSP System)

→ The main elements of DSP system are:-

i) Analog to Digital converter (sampler, Quantizer, Encoder)
ii) Digital signal processor
iii) Digital to analog converter (Interpolator, LPF, Decoder)

→ First the continuous time analog signal is provided to the A to D converter.

→ The sampler converts the continuous time continuous amplitude signal into descrete time continuous continuous signal.

→ The O/P of sampler is fed to quantizer which converts the discrete time continuous amplitude signal to discrete time amplitude signal.

→ The encoder provides a particular value to the quantized signal so that we get the digital signal $x(t)$ at the A to D converter O/P.

→ The O/P of A to D converter is digital signal which is fed to digital signal processor.

→ Digital signal processor may be a large programmable digital computer or may be a hardware digital processor to perform a specific set of operations

→ The output of digital signal processor is digital in nature so we ~~again~~ have to again convert it to analog signal by the help of D to A converter.

→ In D to A converter, a proportional output is produced to that of the binary values.

→ ~~In~~ The interpolator converts the analog samples into analog signal.

→ The low pass filter attenuates the undesirable high frequency components.

Advantages of DSP over Analog Signal Processing

→ Digital Signal Processing have the following advantage over analog signal processing :-

i) Flexibility :- Digital Signal Processing (DSP) operations by changing the program in the digital signal processor which is difficult in case of analog signal processor.

ii) Greater Accuracy :- Digital Signal Processing (DSP) have greater accuracy and control over analog signal processing

iii) Data Storage is Easy :- Digital signal can be stored easily on CD, drives and any storage devices

and it can reproduce easily which is not possi
in case of analog signal processing.

→ DSP is less sensitive to environmental changes an
tolerance to component value.

→ DSP can process very low frequency signal.

→ DSP allows time sharing.

Disadvantages of DSP

→ System complexity
→ Limited bandwidth

Applications of DSP

→ Speech Recognition
→ Image processing
→ RADAR signal processing
→ Spectrum Analysis
→ SONER signal processing

Classification of Signals

Broadly signals are classified into two types —
i) continuous time signal — $x(t)$
ii) Discrete time signal — $x(n)$

Signals can be further classified in the following type —
a) Deterministic and Non-deterministic
b) Periodic and Aperiodic signal
c) Even signal and odd signal
d) Energy signal and Power signal
e) Multi-channel signal and Multi-dimensional signal
f) Continuous value signal and Discrete value signal

Multi-Channel Signal

→ If the signal is generated by a single source then it is called as single channel signal.

→ Signals which are generated from more than one source are known as multi-channel signal

→ These signals are represented by a vector.

Example:- $S(t) = \begin{bmatrix} S_1(t) \\ S_2(t) \\ S_3(t) \end{bmatrix}$

$S(t)$ is a 3-channel signal

* The ground accelerations due to earthquake.
* In ECG it may be either 3-lead so that the signals may be 3-channel or 12 channel.

Multi-Dimensional Signal

A signal is said to be multi-dimensional if it is a function of more than one variable.

Example:-

* Picture is a 2-dimensional signal.
* colour TV signal is a 3-dimensional signal.

Continuous Time Signal

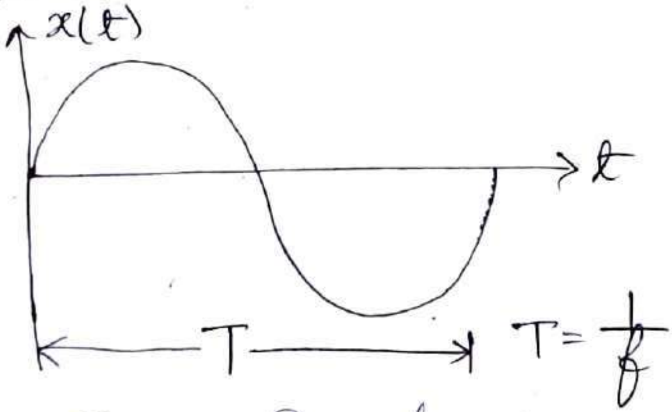
→ The signal that varies with time is known as continuous time signal.

→ It is defined for each value of time.

→ It takes the value in continuous interval, from $-\infty$ to $+\infty$.

Example:- $x_1(t) = \sin \pi t$

* A continuous time signal is represented by $x(t)$ where 't' is the independent variable.



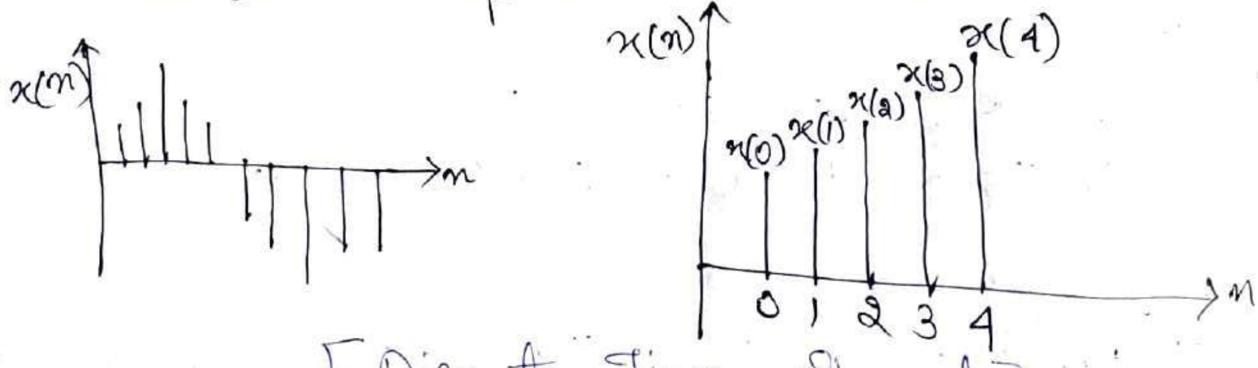
Discrete Time Signal

→ A discrete time signal is defined only at certain time instant.

→ This time constant may be or may not be equi-distant but usually taken at equally spaced interval.

→ In discrete time signal the amplitude between two time instant is not defined.

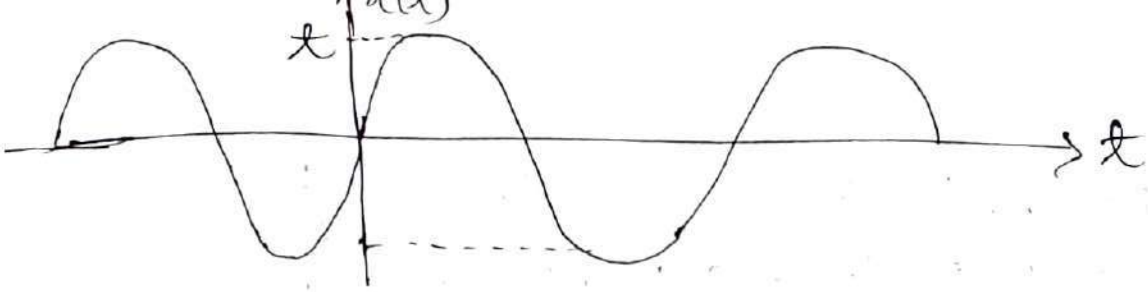
→ A discrete time signal is denoted by $x(n)$ where 'n' is the independent variable.



[Discrete Time Signal]

Continuous Value Signal

→ A signal is said to be continuous valued signal if it takes all possible value on a finite or infinite range.



Discrete Value Signal

→ Discrete valued signal takes some values from a given set.

→ Normally, these values are equidistant and can be expressed in an integer multiple of the distance between two successive values.

Concept of Frequency in Continuous Time Sinusoidal Signal

Mathematically, a continuous time sinusoidal signal can be represented by

$$\cancel{x(a)}\ \boxed{x_a(t) = A \sin(\Omega t + \theta); \ -\infty \leq t \leq \infty}$$

where, x_a = Analog signal

A = Amplitude

Ωt = frequency in rad/sec

θ = phase

$$\boxed{\Omega = 2\pi F}$$

$$\boxed{F = \frac{1}{T}}$$

T = Time period

Concept of Frequency in Discrete Time Sinusoidal Signal

Mathematically, it is represented by:

$$\boxed{x(n) = A\sin(\omega t + \theta);\ -\infty \leq n \leq \infty}$$

where, $x(n)$ = Discrete time signal

A = Amplitude

ω = Frequency in rad/sec

θ = phase in radian

$\omega = 2\pi F$

Property of Analog Signal (Continuous Time Signal):

Property-1:-

If $x_a(t) = x_a(t+T)$ then $x_a(t)$ is periodic with time period (T).

Property-2:-

Continuous time sinusoidal signal with different frequencies are themselves different.

Property-3:-

Increasing the frequency results in an increase in the rate of oscillation of the signal.

Property of Discrete Time Signal

Property-1:-

Discrete Sinusoides are periodic only if its frequency 'F_0' is a rational number.

Harmonically Related Complex Exponential

Continuous Time Exponential :-

The basic signal for continuous time harmonically related exponential at :

$$\boxed{S_K(t) = e^{jK\Omega_0 t}}$$

where, $K = \text{Integer}$
$= 0, \pm 1, \pm 2, \ldots\ldots$

The '$x_K(t)$' can be written as in the form of harmonically related complex exponential.

$$x_a(t) = \sum_{K=-\infty}^{\infty} C_K S_K(t) = \sum_{K=-\infty}^{\infty} C_K e^{jK\Omega_0 t}$$

Discrete Time Exponential :-

$$S_K(n) = e^{jK\omega_0 n}, \quad K = 0, \pm 1, \pm 2, \ldots$$

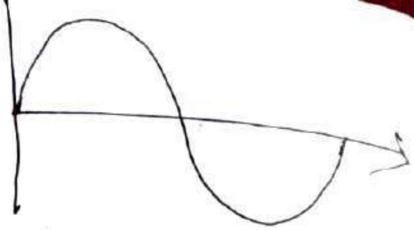
$$x(n) = \sum_{n=0}^{n-1} C_K S_K(n) = \sum_{n=0}^{n-1} C_K e^{j2\pi \frac{n}{N} K}$$

Sampling Of Analog Signal

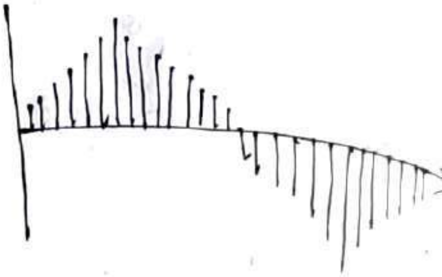
There are many ~~ways~~ ways to sample an analog signal but we will consider only the periodic sampling or uniform sampling.

Here, $x(n)$ is the discrete time signal which is obtained by sampling of analog signal $x(t)$ at every T sec.

$x_a(n) = x_a(nT)$



Sampler



→ Here, T is the time interval between the successive samples known as sampling interval.

→ The relation between t and n, $t = nT$. Similarly the relation between Ω and ω.

Let us consider an analog signal $x_a(t)$.

$$\boxed{x_a(t) = A\cos(2\pi Ft + \theta)}$$

After sampling with a frequency $F_s = \frac{1}{T}$

$$x_a(t) = A\cos(2\pi Ft + \theta)$$
$$= A\cos(2\pi FnT + \theta)$$
$$= A\cos\left(2\pi \frac{F}{F_s} \cdot n + \theta\right)$$
$$= A\cos(2\pi fn + \theta)$$

$$\boxed{f = \frac{F}{F_s}}$$

F = continuous
f = discrete

$$\boxed{\omega = \frac{\Omega}{F_s}}$$

For continuous time sinusoidal signal, the frequency F and Ω

$-\infty < F \infty$

$-\infty < \Omega < \infty$

For discrete time sinusoidal signal the range of frequency variable

$$(\omega) = -\pi < \omega < \pi$$
$$f = -\pi/2 < F < \pi/2$$

So, we conclude that the frequency of continuous time sinusoidal signal when sampled at a rate of $F_S = \frac{1}{T}$ must be in the range of $-\pi/2$ and $\pi/2$.

Sampling Theorm

Sampling theorm states that "a signal can be exactly reproduced if it is sampled at a frequency F_S, where F_S is greater than twice the maximum frequency in the signal."

i.e, $\boxed{F_S \geqslant 2 f_m}$

Here, F_S = Sampling Frequency

f_m = maximum frequency present in the signal.

→ When sampling rate is exactly equal to $2f_m$ then the sampling rate is known as nyquist rate.

→ Nyquist rate is also called as minimum sampling rate which is represented by:

$$\boxed{F_S = 2f_m.}$$

→ Maximum Sampling interval is known as Nyquist interval, $T_s = \frac{1}{2f_m}$

* Interpolation :- The process of reconstructing a continuous time signal $x(t)$ from its ~~sampled~~ samples is known as interpolation.

Quantization of Continuous Amplitude Signal :-

→ A digital signal is a sequence of number in which each number is by a finite number of digits.

→ The process of converting a discrete time continuous amplitude signal into a discrete time discrete amplitude signal by expressing each sample value as a finite number of digits is called Quantization.

→ In this process some errors are generated which is called Quantization error.

→ If the input of quantizer is $x(n)$ and output of quantizer is $x_q(n)$ then we can define quantization error as the difference of quantized value and sampled value.

$$\boxed{e_q(n) = x_q(n) - x(n)}$$

Coding of Quantized Sample :-

In coding process a unique binary number is assigned to each quantization level.

→ If we have n levels then we need atleast n different binary number ~~length~~ length of 'b' no. of bits and we can create 2^b binary number.

Q.) Consider an Analog ~~signal~~ Signal $x_a(t) = 3\cos 100\pi t$.

a) Determine the minimum Sampling rate to avoid aliasing.

b) Suppose the signal is sampled at a rate of ~~$F_s = 200H$~~ $F_s = 200Hz$, then what is the discrete time signal after sampling?

Ans— $x_a(t) = 3\cos 100\pi t$ ($\because A\cos\omega t = A\cos 2\pi f t$)

$$A = 3$$
$$f_m = 50Hz$$

a) Minimum sampling rate/Nyquist Rate

$$F_s = 2f_m$$
$$\Rightarrow F_s = 2\times 50 = 100Hz$$

b) $A\cos \frac{F}{F_s} n$

$$x(n) = A\cos 2\pi \frac{F}{F_s} n$$
$$= 3\cos \frac{100\pi}{200} n$$
$$\Rightarrow \boxed{x(n) = 3\cos \frac{\pi}{2} n}$$

Q.) Consider an analog Signal $x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$. Then find out the nyquist rate for the particular Signal.

Ans - $3\cos 50\pi t = A\cos 2\pi f_1 t$

$$2f_1 = 50$$
$$\Rightarrow f_1 = 25Hz$$

$10\sin 300\pi t = A\cos 2\pi f_2 t$

$$2f_2 = 300$$
$$\rightarrow f_2 = 150 Hz$$

$\cos 100\pi t = A\cos 2\pi f_3 t$

$$2f_3 = 100$$
$$\Rightarrow f_3 = 50Hz$$

$\therefore$ Here, $f_m = 150Hz$

$F_S = 2f_m = 2\times 150 = 300Hz.$

Digital to Analog Conversion

To convert a digital signal to analog signal D to A converter is used.

→ The function of D to A converter is to interpolate the samples.

→ Sampling theorm specifies the optimum interpolation for a band limited signal.

* A zero order hold is the simple type of DAC.

Analysis of Digital System Signals Vs Discrete time Signal System

We know that the values of digital signal are taken from a finite set of possible values. This signal is used in digital computer.

→ Generally computer operate on the number which are represented by string of zero and ones.

The length of this string is fixed and finite. Sometimes this finite word length cause communication in the analysis of DSP system.

To avoid this complication we neglate the quantized nature of digital signal and system in various case and consider discrete time signal and system.

Assignment - 1

1) What is signal processing? Draw the block diagram and explain the signal processing system / elements of DSP? (10)

2) Describe the advantages of digital signal processing over analog signal processing. (5)

3) State sampling theorm. (2)

4) Draw and Explain the principle of analog to digital converter. (5)

5) What is quantization? (2)

6) What is multi-channel and multi-dimensional signal? (2)

7) Classify different signal and explain briefly. (10)

8) What are the applications of signal processing. (2)

9) Differentiate discrete signal and digital signal. (2).

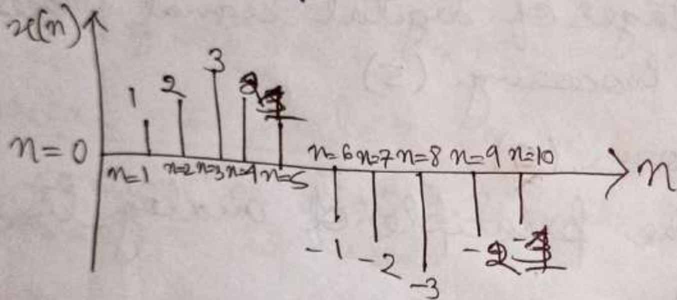
II → Discrete Time Signals and Systems

A discrete time signal $S(n)$ is a function of an independent variable 'n'.

Discrete time signal can be represented in the following 4 ways:

1. Graphical Representation
2. Functional Representation
3. Tabular Representation
4. Sequential Representation

① Graphical Representation



② Functional Representation

$$x(n) = \begin{cases} -1 & \text{for } n = 10 \\ -3 & \text{for } n = 8 \\ -1 & \text{for } n = 6 \\ 2 & \text{for } n = 4 \\ 2 & \text{for } n = 2 \\ 1 & \text{for } n = 1 \end{cases}$$

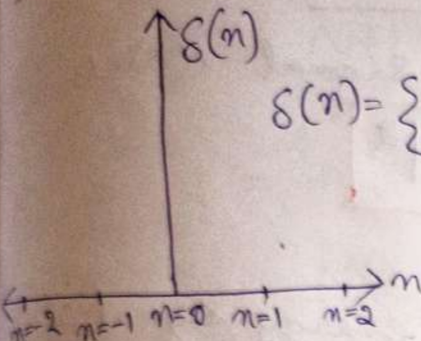
③ Tabular Representation

n	1	2	3	4	5	6	7	8	9	10
$x(n)$	1	2	3	2	1	-1	-2	-3	-2	-1

① **Sequential Representations**

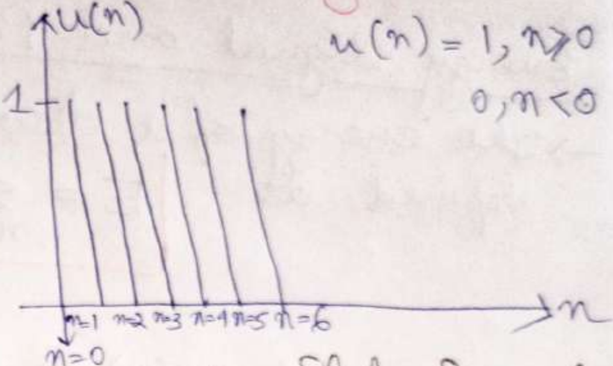
$x(n) = \{0, 1, 2, 3, \cancel{2}, \cancel{\ }\ 1, -1, -2, -3, -2, -1\}$

Some Elementary Discrete Time Signals :-



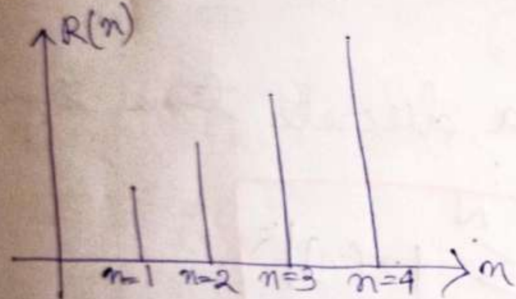
$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Unit Impulse Signal



$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Unit Step Signal



$$R(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

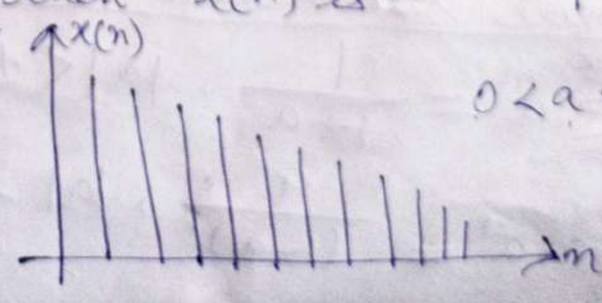
Unit Ramp Signal

Unit Exponential Signal

→ An exponential signal is represented by $x(n) = a^n$ for all values of 'n'.

If 'a' is real then $x(n)$ is real.

If 'a' is a complex value then $x(n)$ is a complex value function



when 'a' is complex i.e., $a = re^{J\theta}$

then $x(n) = r^n e^{J\theta n}$

$= r^n(\cos\theta n + J\sin\theta n)$

Classification of discrete time Signal

Energy Signal and Power Signal

→ The energy of a discrete time signal $x(n)$ is defined as

$$\boxed{E = \sum_{n=-\infty}^{\infty} |x(n)|^2}$$

→ A signal $x(n)$ is called as energy signal if the energy is finite $(0 < E < \infty)$ and power equal to zero.

→ The average power of a discrete time signal $x(n)$ is defined as:

$$\boxed{P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2}$$

* $0 < P < \infty$
* $E = \infty$

→ The signal $x(n)$ is called as power signal if and only if the power is finite and energy is infinite.

Infinite Summation Formula

① $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$, $|a| < 1$

② $\sum_{n=0}^{\infty} n a^n = \frac{a}{(1-a)^2}$, $|a| < 1$

③ $\sum_{n=0}^{\infty} n^2 a^2 = \frac{a^2+a}{(1-a)^3}$

Finite Summation Formula:—

① $\sum_{n=0}^{N} a^n = \frac{1-a^{n+1}}{a}$, $a \neq 1$

② $\sum_{n=0}^{N} 1 = N+1$

③ $\sum_{n=N_1}^{N_2} 1 = N_2 - N_1 + 1$

④ $\sum_{n=0}^{N} n = \frac{N(N+1)}{2}$

⑤ $\sum_{n=0}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$

Q.) Check the power and energy of the unit step signal.

Ans— $x(n) = u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} |u(n)|^2$$

$$= \sum_{n=0}^{\infty} (1)^2 = \sum_{n=0}^{\infty} 1 = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |u(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \times N+1$$

$$= \lim_{N \to \infty} \frac{N+1}{2N+1}$$

$$= \lim_{N \to \infty} \frac{1+\frac{1}{N}}{2+\frac{1}{N}} = \frac{1}{2}$$

Therefore, the unit step signal is a power signal. Its energy is infinite

Periodic Signal AND Aperiodic Signal

→ A signal $x(n)$ is periodic with N when
$x(n+N) = x(n)$ for all values of n —— (1)

* The smallest value of N for which eqn. (1) is valid is known as fundamental period

→ If eqn. (1) is not satisfied for any value of 'N' then the signal is aperiodic signal.

We know that $\omega_0 N = 2\pi K$

$$\Rightarrow \omega_0 = \frac{2\pi K}{N}$$

$$\Rightarrow \cancel{2\pi} f_0 = \frac{\cancel{2\pi} K}{N}$$

$$\Rightarrow \boxed{f_0 = \frac{K}{N}}$$

Q.) Determine whether the following signals are periodic or not and also find the fundamental period.

(1) $x(n) = \cos 0.01\pi n$

$= \cos \omega_0 n$

$$N = \frac{2\pi K}{\omega_0}$$

$$= \frac{2\pi K}{0.01\pi} = \frac{2K}{0.01} = \frac{2}{\frac{1}{100}} \times K$$

$$= 200K$$

$$\Rightarrow \boxed{N = 200K}$$

* The signal is periodic because it is multiple of 2π.
* If minimum value of K is taken then fundamental period is 200.

② $x(n) = \sin(\pi/4)n$, $K, N = ?$

$= \sin \omega n$

$\omega = \pi/4$

$\Rightarrow 2\pi f = \pi/4$

$\Rightarrow f = \frac{1}{8} = \frac{K}{N}$

$\therefore K = 1$ and $N = 8$.

③ $x(n) = \cos 2\pi/3\ n + \sin \pi/2\ n$

$\cos 2\pi/3\ n$

$\omega = 2\pi/3$

$\Rightarrow \cancel{2\pi} f = \cancel{2\pi}/3$

$\Rightarrow f = \frac{1}{3} = \frac{K}{N_1}$

$K = 1, N_1 = 3$

$\sin \pi/2\ n$

$\omega = \pi/2$

$\Rightarrow \cancel{2\pi} f = \cancel{\pi}/2$

$\Rightarrow f = \frac{1}{4} = \frac{K}{N_2}$

$K = 1, N_2 = 4$

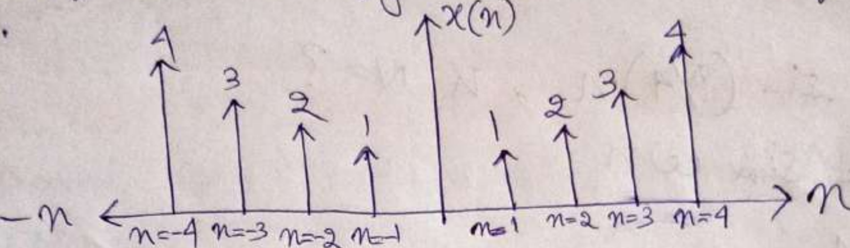
$\therefore N = LCM(N_1, N_2) \Rightarrow N = 12.$

* A signal is periodic if $F = \frac{K}{N}$ is a rational number.

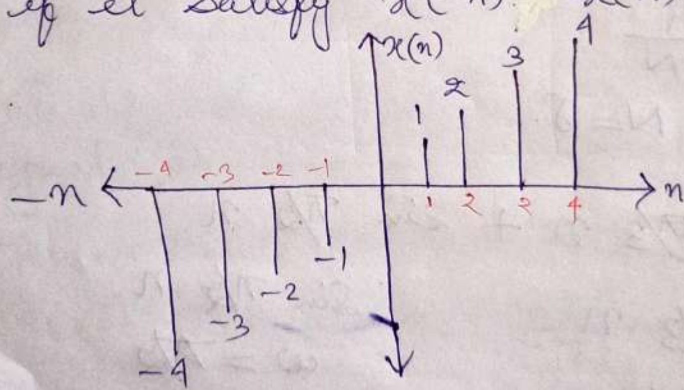
* All periodic signals are power signals.
* All aperiodic signals are energy signals.
* Ramp signals are neither power nor energy signals

Symmetric (Even) and Anti-Symmetric signals

→ A discrete time signal is said to be symmetric (even) if it satisfy $x(-n) = x(n)$ for all values of n.



→ A discrete time signal is said to be anti-symmetric (odd) if it satisfy $x(-n) = -x(n)$ for all values of n.



* Even signal is symmetric above x-axis and odd signal is symmetric about origin.

Simple Manipulation of Discrete Time Signals

1) **Shifting**:- The shifting operations shift the values of the sequence by an integer variable.

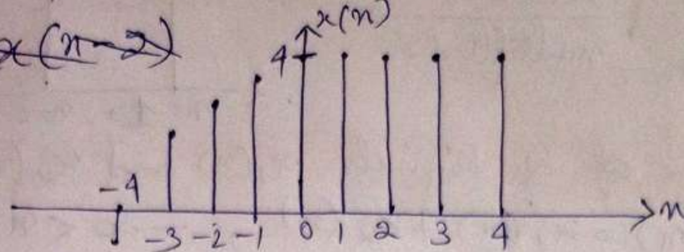
→ Shifting can be either delay or advance.

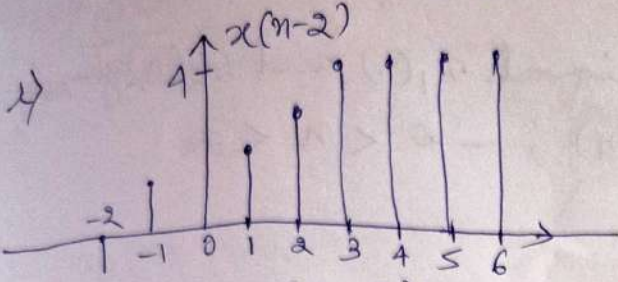
→ Shifting is represented by:

$$y(n) = x(n-k)$$

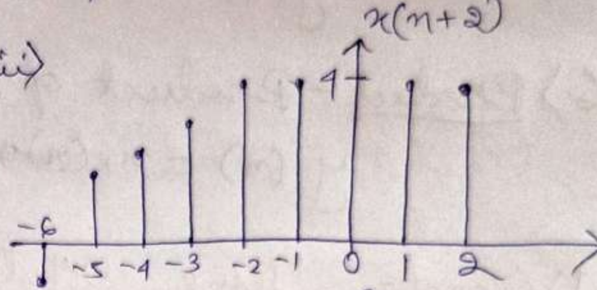
If $K = +ve \rightarrow$ delay
$K = -ve \rightarrow$ advance

Ex- y ~~$(n) = x(n-2)$~~



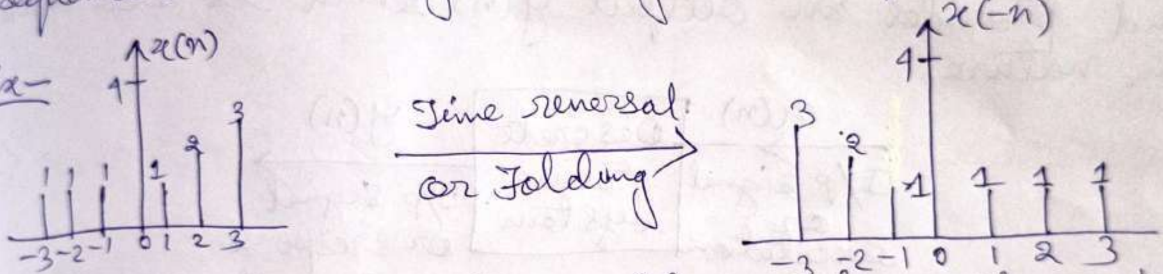
i) 

$y(n) = x(n-2)$

ii) 

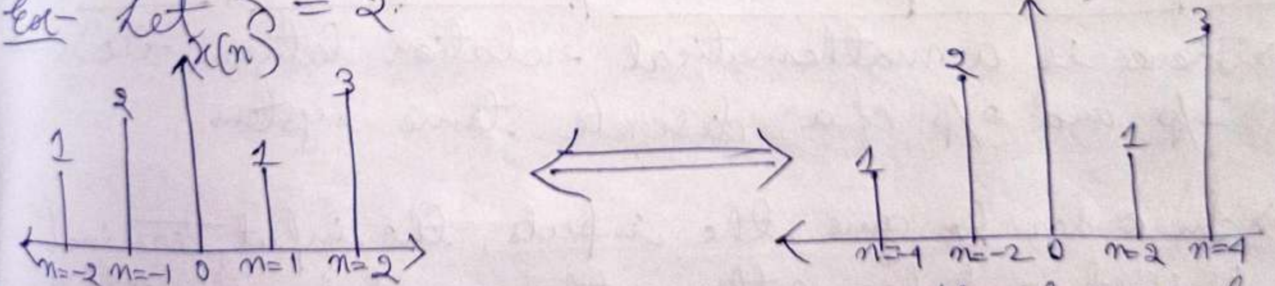
$y(n) = x(n+2)$

2.) <u>Time Reversal (Folding)</u> :- Time reversal of a sequence is done by holding the sequence about $n=0$

Ex- 

3.) <u>Time Scaling</u> :- It is obtained by replacing 'n' by 'λn'.

Ex- Let $\lambda = 2$.



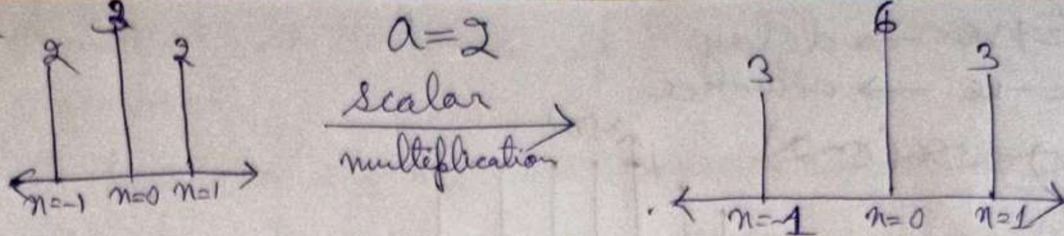
Before time scaling

After time scaling.

4.) <u>Scalar Multiplication</u>

$x(n) \xrightarrow{\text{Scalar multiplication}} y(n) = a\, x(n)$

Ex-



5.) **Sum**:- Sum of 2 signals $x_1(n)$ and $x_2(n)$ is given by

$$y(n) = x_1(n) + x_2(n), \quad -\infty < n < \infty$$

6.) **Product**:- Product of 2 signals $x_1(n)$ and $x_2(n)$ given by

$$y(n) = x_1(n)\, x_2(n), \quad -\infty < n < \infty$$

Discrete Time System

A discrete time system is a device that operates on a discrete time input signal $x(n)$, process the input signal with some definite functions and provides an output $y(n)$ which is also discrete in nature.

$$y(n) = T[x(n)]$$

* where T denotes transformation.

Input-Output Description of a System

-> There is a mathematical relation between the I/p and o/p of a discrete time system

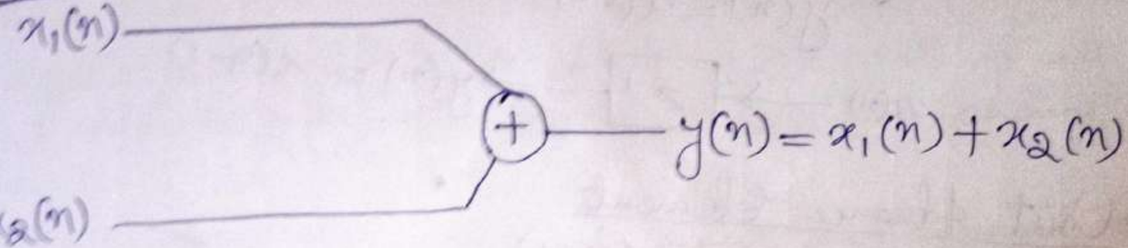
-> In order to give the inputs, the input terminal is used and for getting the outputs, the output terminal is used.

-> The input and output both must be discrete in nature.

→ We can express $x(n) \xrightarrow{T} y(n)$, which means $y(n)$ is the response of the system and $x(n)$ denotes the excitations.

Block diagram representations of discrete-time systems

⟹ **Adder**



Example – a) $x_1(n) = \{\underset{\uparrow}{1}, 2, 1, 2\}$

$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$

$y(n) = \{\underset{\uparrow}{2}, 4, 4, 6\}$

b) $x_1(n) = \{\underset{\uparrow}{1}, 2, 1, 2\}$

$x_2(n) = \{2, \underset{\uparrow}{1}, 3, 4\}$

$= \{2, \underset{\uparrow}{2}, 5, 5, 2\}$

⟹ **Constant Multiplier**

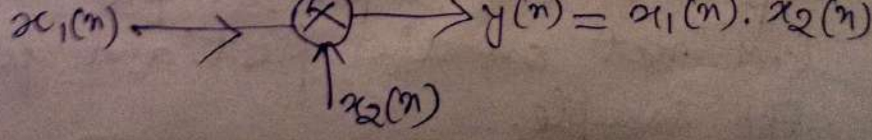
$$x(n) \longrightarrow a\,x(n)$$

$x(n) = \{2, 4, 6\}$

$a = 2$

$a\,x(n) = \{4, 8, 16\}$

⟹ **Multiplier**



$$x_1(n) = \{\underset{\uparrow}{2}, 3, 4, 5\}$$

$$x_2(n) = \{1, \underset{\uparrow}{2}, 3, 4\}$$

$$y(n) = \{0, 4, 9, 16, 0\}$$

→ Unit Delay Element

$$y(n) = x(n-1)$$

$$x(n) \longrightarrow \boxed{z^{-1}} \longrightarrow y(n) = x(n-1)$$

⟹ Unit Advance Element

$$y(n) = x(n+1)$$

$$x(n) \longrightarrow \boxed{z} \longrightarrow y(n) = x(n+1)$$

Classification of Discrete Time System :-

→ Discrete time systems are classified depending on its various properties and characteristics as follows

1.) Static and dynamic System
2.) Time variant and Time invariant System
3.) Causal and non-causal System
4.) Linear and non-linear System
5.) FIR and IIR Response
6.) Stable and Unstable System

Static and Dynamic System —

→ A ~~static~~ discrete time system is called static or memory less system if the output at any instant depends on the input sample at same instant, but not on the past and future values of the sample.

→ A system having memory is called as dynamic system.

Example – i) $y(n) = 2x^3(n) \rightarrow$ Static System

ii) $y(n) = 5x(n-1) + 2x(n-2) \rightarrow$ Dynamic System

Time Variant and Time Invariant System

→ A System is said to be time invariant if its input and output characteristics does not change with time.

→ If the response to a delayed input and delayed response are equal then, the system is said to be time invariant.

→ Let the input of the System is $x(n)$ and output of the system is $y(n)$.

The response to a delayed input is denoted by

$$\boxed{y(n, K) = T[x(n-K)]}$$

→ Here the input is delayed by K samples now the output is delayed by K samples, So, $y(n-K)$

→ If $y(n,K) = y(n-K)$ for all values of K then the System is time invariant. $(\Rightarrow x(n-K) \Rightarrow y(n-K))$

→ If $y(n,K) \neq y(n-K)$ for all values of K then the system is time variant.

Q) Test whether the system is time variant or invariant.

i) $y(n) = x(n) - x(n-1)$

Ans $y(n,K) = x(n-K) - x(n-K-1)$

$y(n-K) = x(n-K) - x(n-K-1)$

$\Rightarrow y(n,k) = y(n-k)$

So, the system is time-invariant.

ii) $y(n) = x(-n)$

The response of this system to $x(n-k)$ is

$$y(n,k) = x(-n-k)$$

If we delay the output $y(n)$ by k units in time, the result will be:

$$y(n-k) = x(-n+k)$$

$\because y(n,k) \neq y(n-k)$, the system is time variant

iii) ~~y(n) = n(x~~ $y(n) = nx(n)$

$$y(n,k) = nx(n-k)$$

If we delay $y(n)$ by k units in time, the result will be:

$$y(n-k) = (n-k)\,x(n-k)$$

$$= nx(n-k) - kx(n-k)$$

$\therefore$ The system is time variant as $y(n-k) \neq y(n,k)$.

Causal and Non-causal System—

→ A system is said to be causal if the output depends on the present and past input of the sample.

→ If the output of the system depends on future input along with present and past inputs, then the system is said to be non-causal system.

Example— 1) $y(n) = x(n) + \dfrac{1}{x(n-1)}$

Let $n = 0$.

$$y(0) = x(0) + \frac{1}{x(0-1)}$$
$$= x(0) + \frac{1}{x(-1)}$$

let $n = 1$

$$y(1) = x(1) + \frac{1}{x(1-1)} = x(1) + \frac{1}{x(0)}$$

So, the system is causal as the output depends on past and present inputs only.

2) $y(n) = x(n^2)$

let $n = 0$

$$y(0) = x(0^2) = x(0)$$

let $n = 1$

$$y(1) = x(1^2) = x(1)$$

let $n = 2$

$$y(2) = x(2^2) = x(4)$$

Hence, the system is non-causal as the output value depends on present, past, as well as future input values

Linear and Non-Linear Systems

A system that satisfy the principle of superposition is said to be a linear system

* Superposition principle states that "the response of a system to a weighted sum of signal is equal to the corresponding weighted sum of output signal to each of individual output."

$a_1 x_1(n) \longrightarrow a_1 y_1(n)$

$T[a_1 x_1(n) + a_2 x_2(n)] \longrightarrow T[a_1 x_1(n)] + T[a_2 x_2(n)]$

or, for a linear system

$T[a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)]$

→ The system which does not follow the principle of superposition is called a non-linear system.

Q.) Test whether the following system is linear or non-linear.

1.) $y(n) = x^2(n)$

$y_1(n) = x_1^2(n)$

$y_2(n) = x_2^2(n)$

$a_1 y_1(n) = a_1 x_1^2(n)$

$a_2 y_2(n) = a_2 x_2^2(n)$

$y(n) = a_1 x_1^2(n) + a_2 x_2^2(n) + \ldots\ldots$ (1)

$y(n) = [a_1 x_1(n) + a_2 x_2(n)]^2 \ldots\ldots$ (2)

Eqn. (1) $\neq$ (2), so the system is non-linear.

2.) $y(n) = n\,x(n)$

$y_1(n) = n\,x_1(n)$

$y_2(n) = n\,x_2(n)$

$y(n) = a_1 n x_1(n) + a_2 n x_2(n) \ldots\ldots$ (1)

$T[a_1 x_1(n) + a_2 x_2(n)]$

$= a_1 n x_1(n) + a_2 n x_2(n) \ldots\ldots$ (2)

Eqn. (1) = (2), ∴ the system is linear.

Stable System and Unstable System

A system is said to be stable if it produces bounded output for bounded input.

→ The ~~nece~~ necessary conditions for stability is summession.

$$\sum_{n=-\infty}^{\infty} h(n) < \infty$$

Q.) Test the stability of the system whose impulse response is $h(n) = \left(\frac{1}{2}\right)^n u(n)$.

$$\sum_{n=-\infty}^{\infty} h(n)$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n)$$

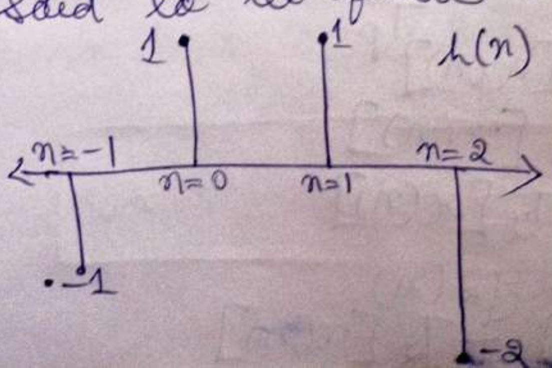
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cdot 1$$

$$= \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{2-1}{2}} = 2$$

As $\sum_{n=-\infty}^{\infty} h(n) < \infty = 2$, so the system is stable.

Finite Impulse Response (FIR) and Infinite Impulse Response (IIR)

→ If the impulse response is of finite duration the system is said to be finite impulse response system



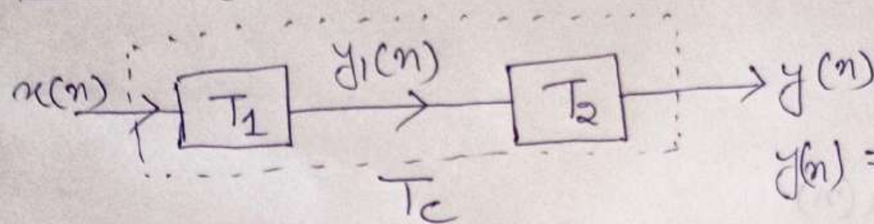
$$h(n) = \{1,\ n = 0, 1 \quad -2,\ n = 2 \quad -1,\ n = -1\}$$

→ An infinite impulse response system have infinite duration sequence

Interconnection of Discrete - Time System

Interconnections of D.T systems are of two types:

1.) Cascade (series) →

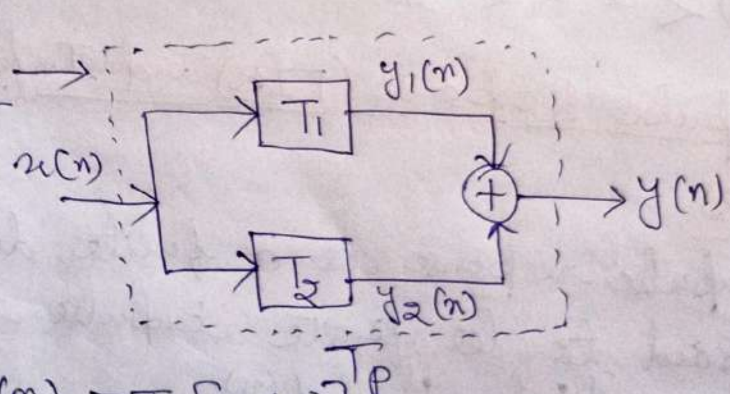


$y(n) = T_c[x(n)]$

$$y_1(n) = T_1[x(n)]$$
$$y(n) = T_2[y_1(n)]$$
$$= T_2[T_1[x(n)]]$$

$\boxed{T_c \equiv T_1 T_2}$, But $\boxed{T_2 T_1 \neq T_1 T_2}$

2.) Parallel →



$$y(n) = T_p[x(n)]$$
$$y_1(n) = T_1[x(n)]$$
$$y_2(n) = T_2[x(n)]$$
$$y(n) = y_1(n) + y_2(n)$$
$$= T_1[x(n)] + T_2[x(n)]$$

$$T_p[x(n)] = x(n)[T_1 + T_2]$$

$$\boxed{T_p \equiv T_1 + T_2}$$

Discrete - Time Linear Time - Invariant System

→ LTI - combination of both linear and time invariant systems

linear - which satisfies ~~the~~ superposition principle.

Time invariant - any delay in input will reflect in output.

Techniques for the analysis of linear system

→ There are two basic methods for analyzing the response of a linear system to a given input signal.

1st method → It is based on direct solution of input - output equation.

$$y(n) = F[y(n-1), y(n-2), \ldots, y(n-N), x(n), x(n-1), x(n-2), \ldots x(n-M)]$$

where $F[\cdot]$ → some function of quantities in brackets

Specially for an LTI system:

$$y(n) = -\sum_{K=1}^{N} a_K y(n-K) + \sum_{K=0}^{M} b_K x(n-K)$$

2nd Method → First the input signal is resolved to sum of elementary signals.

* The input signal $x(n)$ is resolved into a weighted sum of elementary signal $\{x_K(n)\}$ so that

$$x(n) = \sum_K c_K x_K(n)$$

where c_K are set of amplitudes (weighting coefficients) signal. Now, the response of the system to elementary signal component $x_K(n)$ is $y_K(n)$.

$$y_K(n) \equiv T[x_K(n)]$$

$$y(n) = T[x(n)] = T\left[\sum_K c_K x_K(n)\right]$$

$$= \sum_K c_K T[x_K(n)]$$

$$= \sum_K c_K y_K(n)$$

Resolution of a Discrete-Time Signal into Impulse

Suppose we have an arbitrary signal $x(n)$ that we wish to resolve into a sum of unit sample sequences.

We select the elementary signal $x_K(n)$ to be:

$$x_K(n) = \delta(n-K)$$

where K represents the delay of the unit sample sequence.

Now suppose that we multiply the two sequences $x(n)$ and $\delta(n-K)$. Since, $\delta(n-K)$ is zero everywhere except at $n=K$, where its value is unity, the result of this multiplication is another sequence that is zero ~~to~~ everywhere except $n=K$, where its value is $x(K)$

$$x(n)\,\delta(n-K) = x(K)\,\delta(n-K)$$

If we repeat this multiplication over all possible delays, $-\infty < K < \infty$, and sum all the product sequences, the result will be a sequence equal to the sequence $x(n)$.

$$\boxed{x(n) = \sum_{K=-\infty}^{\infty} x(K)\,\delta(n-K)}$$

Q.) Consider the special case of a finite-duration sequence given as: $x(n) = \{2, \underset{\uparrow}{4}, 0, 3\}$

Resolve the sequence $x(n)$ into a sum of weighted impulse sequences.

Ans- $x(n) = \{2, \underset{\uparrow}{4}, 0, 3\}$

$n = -1, 0, 1, 2$

$$x(n) = \sum_{K=-\infty}^{\infty} x(K)\,\delta(n-K)$$

$$= \sum_{K=-1}^{2} x(K)\,\delta(n-K)$$

$$= x(-1)\,\delta(n-(-1)) + x(0)\,\delta(n-0) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2)$$

$$= 2\,\delta(n+1) + 4\,\delta(n) + 0\,\delta(n-1) + 3\,\delta(n-2)$$

$$= 2\delta(n+1) + 4\,\delta(n) + 0 + 3\delta(n-2)$$

$$\therefore \boxed{x(n) = 2\delta(n+1) + 4\,\delta(n) + 3\delta(n-2)}$$

Response of LTI Systems to Arbitrary Inputs: Convolution Sum

Response of LTI Systems to Arbitrary Inputs:

Convolution Sum

Having resolved an arbitrary input signal $x(n)$ into ~~an~~ a weighted sum of impulse, now ready to ~~dt~~ determine the response of any relaxed linear system.

We denote the response $y(n, K)$ of the system to the input unit sample sequence $n = K$ by special symbol $h(n, K)$, $-\infty < K < \infty$.

$$y(n, K) \equiv h(n, K) = T[\delta(n-K)]$$

n = time index

K = location of input impulse.

Expression of input expressed as a sum of weighted impulses:
$$x(n) = \sum_{K=-\infty}^{\infty} x(K)\,\delta(n-K)$$

Response of the system to corresponding sum of weighted output is:
$$y(n) = T[x(n)] = T\left[\sum_{K=-\infty}^{\infty} x(K)\,\delta(n-K)\right]$$
$$= \sum_{K=-\infty}^{\infty} x(K)\,T[\delta(n-K)]$$

$$\Rightarrow \boxed{y(n) = \sum_{K=-\infty}^{\infty} x(K)\,h(n,K)} \quad \text{——— (1)}$$

Eq^n. (1) Satisfies the superposition principle.

Now, if we have a time-invariant system
$$h(n) \equiv T[\delta(n)]$$
$$h(n-K) = T[\delta(n-K)]$$

Eq^n. (1) reduces to
$$\boxed{y(n) = \sum_{K=-\infty}^{\infty} x(K)\,h(n-K)} \quad \text{——— (2)}$$

The above formula gives the response $y(n)$ of the LTI system as a function of input signal $x(n)$ and the unit sample (impulse) response $h(n)$ is called a convolution sum.

Suppose that we wish to compute the output of the system at some time instant say $n = n_0$:
$$\boxed{y(n_0) = \sum_{K=-\infty}^{\infty} x(K)\,h(n_0-K)}$$

→ Convolution between $x(K)$ and $h(K)$ consists of following 4 steps:

1) <u>Folding</u>:- Fold $h(K)$ about $K=0$ to obtain $h(-K)$.

2) <u>Shifting</u>:- Shift $h(-K)$ by n_0 to the right if n_0 is positive or to left if n_0 is negative, to obtain $h(n_0-K)$.

3) <u>Multiplication</u>:- Multiply $x(K)$ by $h(n_0-K)$ to obtain the product sequence: $\boxed{V_{n_0}(K) \equiv x(K)\,h(n_0-K)}$

4) <u>Summation</u>:- Sum all the values of the product sequence $V_{n_0}(K)$ to obtain the value of the output at time $n=n_0$.

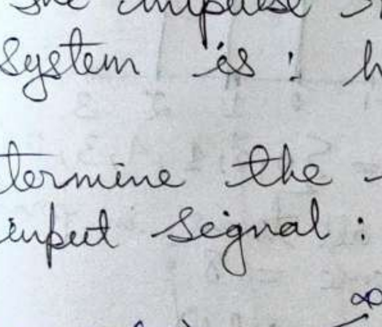
Q.) The impulse response of a linear time-invariant system is: $h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$

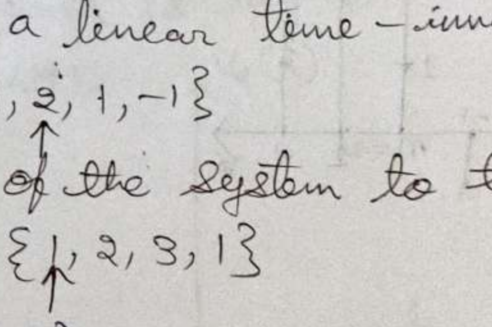
Determine the response of the system to the input signal: $x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$

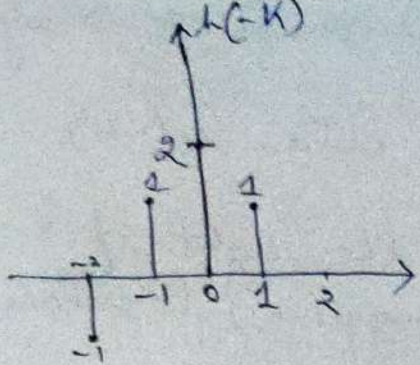
Ans- $$y(n) = \sum_{K=-\infty}^{\infty} x(K)\,h(n-K)$$

→ <u>$n=0$</u>

$$y(0) = \sum_{K=-\infty}^{\infty} x(K)\,h(-K)$$

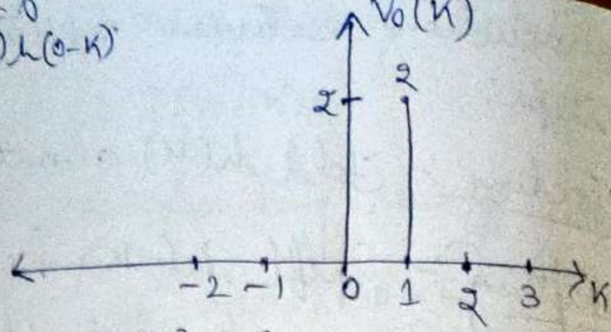






$n = n_0 = 0$

$v_0 = x(K)\,h(0-K)$



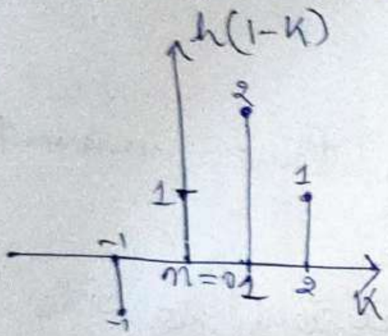
$V_0(K) = \{0, 0, 2, 2, 0, 0\}$

Sum of all terms in product sequence $= 4$.

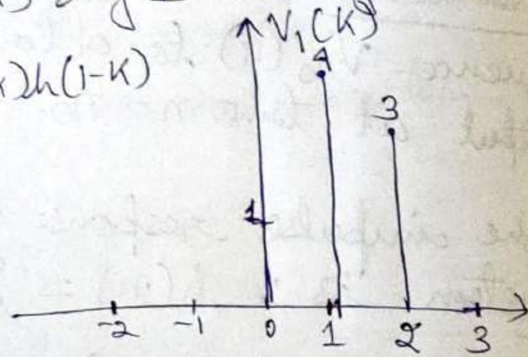
$\rightarrow$ $\underline{n = 1}$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(1-k)$$

$n = n_0 = 1$, Shift $h(-K)$ by 1 to right



$v_1(k) = x(k)h(1-k)$

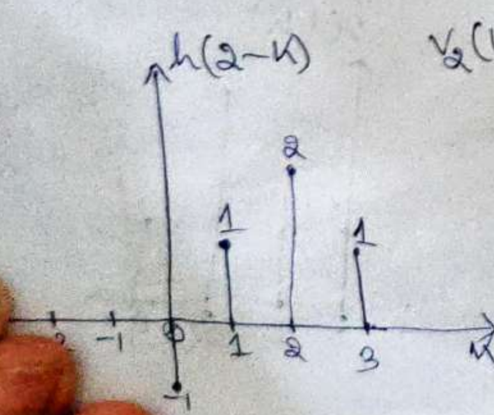


$V_1(k) = \{0, 0, 1, 4, 3, 0\}$

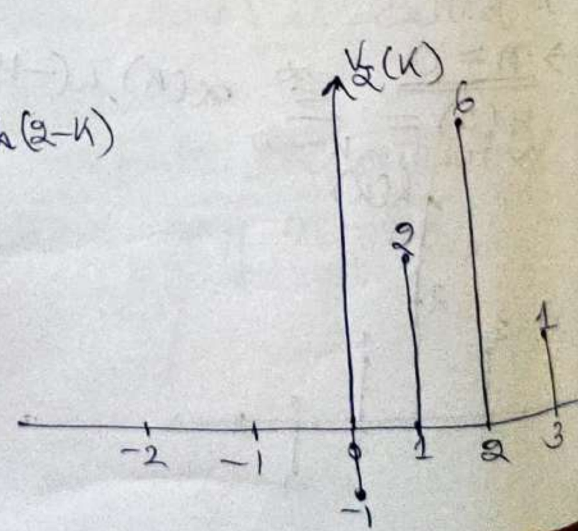
Sum of all terms in product sequence $= 8$

$\rightarrow$ $\underline{n_0 = 2}$ (Shift $h(-K)$ by 2 units to right)

$$y(2) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(2-k)$$



$v_2(k) = x(k) \cdot h(2-k)$



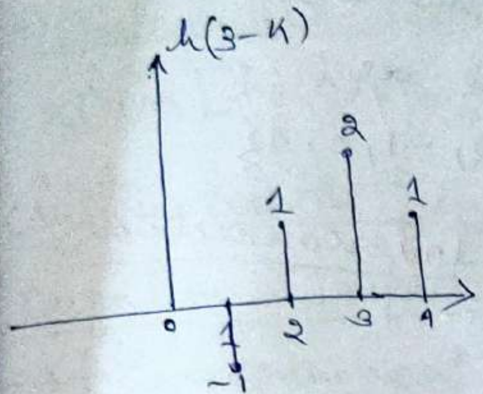
$v_2(K) = \{0, 0, -1, 2, 6, 1\}$

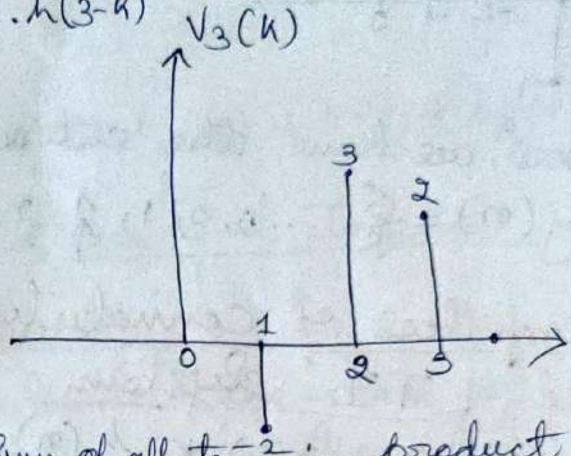
Sum of all terms in product sequence $= 9 - 1 = 8$

$\Rightarrow \underline{n_0 = 3}$ (shift $h(-K)$ by 3 units to right)

$$y(3) = \sum_{K=-\infty}^{\infty} x(K) \cdot h(3-K)$$

$V_3(K) = x(K) \cdot h(3-K)$

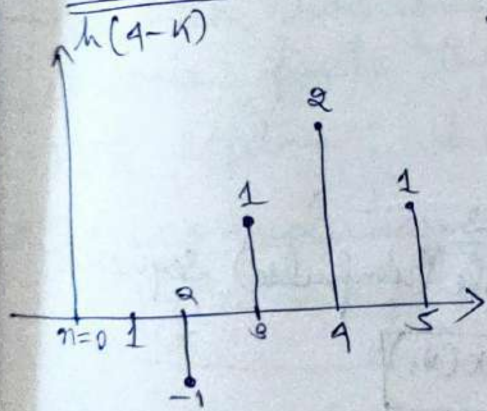


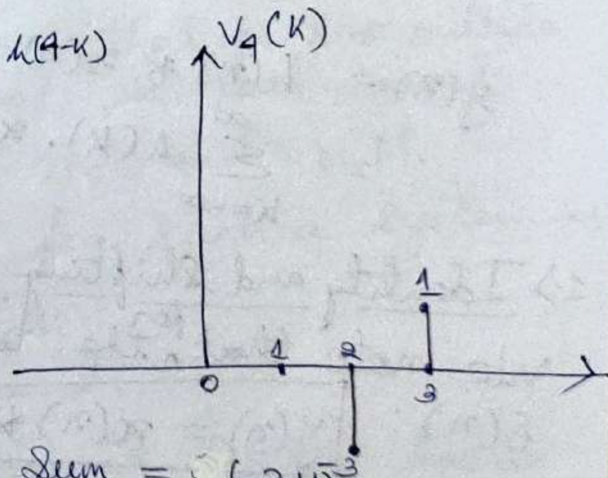


Sum of all terms in product sequence = 3

$\Rightarrow \underline{n_0 = 4}$ (Shift $h(-K)$ by 4 units to right)

$V_4(K) = x(K) \cdot h(4-K)$

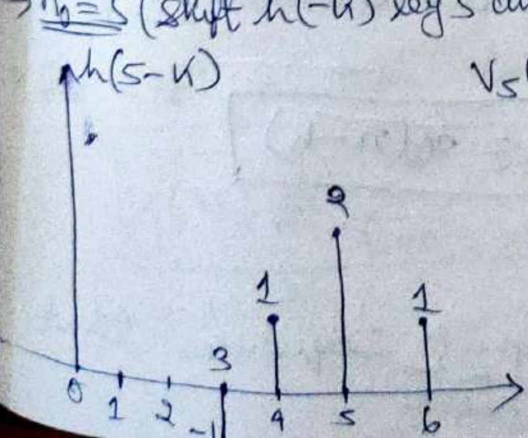


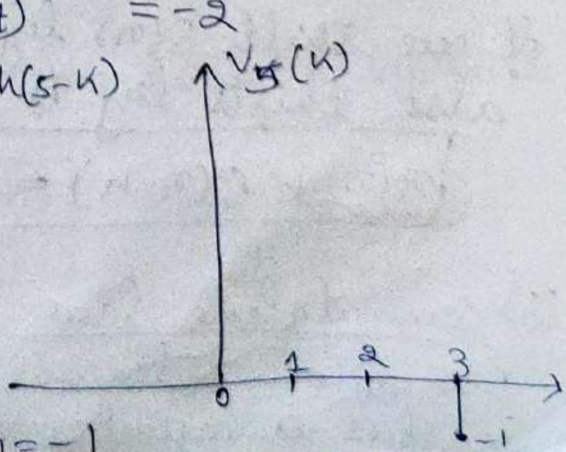


$\therefore$ Sum $= (-3+1) = -2$

$\Rightarrow \underline{n_0 = 5}$ (Shift $h(-K)$ by 5 units to right)

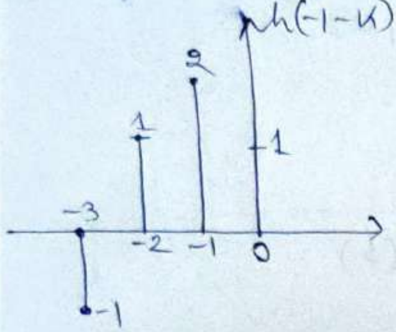
$V_5(K) = x(K) \cdot h(5-K)$



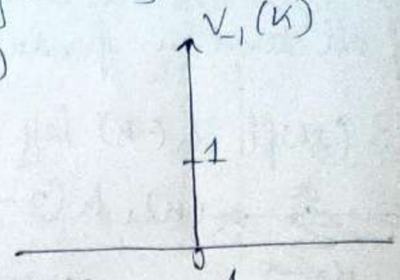


$\therefore$ Sum $= -1$

⟹ $n_0 = -1$ (shift $h(-k)$ to left by 1 unit)



$V_{-1}(k) = x(k) \cdot h(-1-k)$



∴ Sum = 1

Now, we have the entire response as:

$$y(n) = \{\ldots\ldots 0, 0, 1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1, 0, 0\}$$

Properties of Convolution and Interconnection of LTI System

$$y(n) = x(n) * h(n)$$
$$= \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$

$$y(n) = h(n) * x(n)$$
$$= \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k)$$

1.) Identity and Shifting Properties

We note that for the unit sample (impulse) sequence $\delta(n)$: $\boxed{y(n) = x(n) * \delta(n) = x(n)}$

If we shift $\delta(n)$ by k, the convolution sequence also shifts by k units.

$$\boxed{x(n) * \delta(n-k) = y(n-k) = x(n-k)}$$

2.) Commutative law

Operation between two signal sequences that satisfies a number of properties.

$$\boxed{x(n) * h(n) = h(n) * x(n)}$$

3.) Associative law

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

4.) Distributive law

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n).$$

System with Finite Duration and Infinite Duration Impulse Response

→ It is convinient to subdivide the class of linear time-invariant system into two types — finite duration impulse response (FIR) and infinite duration impulse response (IIR).

→ An FIR has impulse response that is zero outside of some finite time interval. So, for causal FIR system: $h(n) = 0$, $n < 0$ and $n \geq M$.

* The convolution formula for such a system is

$$\boxed{y(n) = \sum_{K=0}^{M-1} h(K)\, x(n-K)}$$

→ An IIR linear time invariant system has an infinite duration impulse response.

* The convolution formula for such a system is

$$\boxed{y(n) = \sum_{K=0}^{\infty} h(K)\, x(n-K)}$$

Discrete-Time System Described by Difference Equation

We have treated linear and time-invariant systems that are characterized by their unit sample response $h(n)$. In turn, $h(n)$ allows us to determine the output $y(n)$ of the system for any given input sequence $x(n)$ by means of convolution summation,

$$\boxed{y(n) = \sum_{K=-\infty}^{\infty} h(K)x(n-K)}$$

→ In case of FIR systems, realization involves additions, multiplications, and a finite number of memory locations. So, ~~and~~ an FIR system is readily implemented using convolution sum.

→ In case of IIR systems, practical implementation by convolution is impossible as it requires infinite numbers of memory locations, additions and multiplications. So, IIR discrete-time systems are more conveniently described by difference equations.

Recursive and Non-Recursive Discrete Time System

Suppose we wish to compute the convolution average of a signal $x(n)$ in the interval $0 \leq K \leq n$.

$$y(n) = \frac{1}{n+1} \sum_{K=0}^{n} x(K), \quad n = 0, 1, 2, \ldots$$

the computation of $y(n)$ requires the storage of all the input samples $x(K)$ for $0 \leq K \leq n$.

→ $y(n)$ can ~~be~~ be computed efficiently by utilizing the previous output value $y(n+1)$.

$$(n+1)y(n) = \sum_{k=0}^{n-1} x(k) + x(n)$$

$$= n\,y(n-1) + x(n)$$

$$\Rightarrow \boxed{y(n) = \frac{n}{n+1}\,y(n-1) + \frac{1}{n+1}\,x(n)}$$

→ A System whose present output $y(n)$ depends on any number of past output values is called as Recursive System.

Example:- $y(n) = F\left[x(n),\, y(n-1),\, y(n-2)\right]$

→ A System whose present output $y(n)$ depends on present and past values of Input signal is known as non-recursive System.

Impulse Response of a linear-time Invariant Recursive System

→ For 1st order recursive system, the zero-state response given by :-

$$y_{zs}(n) = \sum_{k=0}^{n} a^k x(n-k)$$

when $x(n) = \delta(n)$

$$y_{zs}(n) = \sum_{k=0}^{n} a^k \delta(n-k)$$

$$= a^n, \quad n \geq 0$$

Hence, $h(n) = a^n u(n)$

For linear-time invariant System

$$y_{zs}(n) = \sum_{k=0}^{n} h(k) \cdot x(n-k), \quad n \geq 0.$$

Q.) Find the convolution of 2 sequences $x_1(n)$ and $x_2(n)$.

$$x_1(n) = \{\underset{\uparrow}{1}, 2, 3\} \qquad x_2(n) = \{\underset{\uparrow}{2}, 1, 4\}$$

Ans- $x_1(n) = \{1, 2, 3\}$ $\quad x_2(n) = \{2, 1, 4\}$

Length of $x_1(n) = L_1 = 3$

length of $x_2(n) = L_2 = 3$

length of $y(n) = L_1 + L_2 - 1 = 3+3-1 = 5$

$x_2(n) \backslash x_1(n)$	1	2	3
2	2	4	6
1	1	2	3
4	4	8	12

$y(n) = x_1(n) * x_2(n)$

$\therefore y(n) = \{\underset{\uparrow}{2}, 5, 12, 11, 12\}$

Q.) Find the convolution of the given sequences:

$$x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

$$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

Ans- length of $x_1(n) = 4$

length of $x_2(n) = 4$

length of $y(n) = L_1 + L_2 - 1 = 7$

$y(n) = x_1(n) * x_2(n)$

$x_2(n) \backslash x_1(n)$	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

$y(n) = \{\underset{\uparrow}{1}, 4, 10, 20, 25, 24, 16\}$

Correlation

correlation is a measure of similarity between two signals.

→ It is of 2 types : i) Auto Correlation

ii) Cross correlation

Auto Correlation

It is defined as correlation of a signal with itself. Auto corelation function is a measure of similarity between a signal and its time delayed version.

Cross-Correlation

cross corelation is the measure of similarity between two different signals.

Q) Find the cross correlation between the given sequences: $x(n) = \{1, 1, 2, 2\}$

$y(n) = \{1, 3, 1\}$

Ans- $x(n) = \{1, 1, 2, 2\}$

$y(-n) = \{1, 3, 1\}$

$x(n)$ \ $y(-n)$	1	3	1
1	1	3	1
1	1	3	1
2	2	6	2
2	2	6	2

$r_{xy}(n) = \{1, 4, 6, 9, 8, 2\}$

Q.) Find the auto correlation of the given Sequence: $x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$

Ans– $x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$

$x(-n) = \{4, 3, 2, \underset{\uparrow}{1}\}$

$x(n)$ \ $x(-n)$	4	3	2	1
1	4	3	2	1
2	8	6	4	2
3	12	9	6	3
4	16	12	8	4

$r_{xx}(n) = \{4, 11, 20, \underset{\uparrow}{30}, 20, 11, 4\}$

Q.) Find the convolution and correlation of the given Sequences: $x_1(n) = \{\underset{\uparrow}{3}, 1, 2, 1\}$

$x_2(n) = \{\underset{\uparrow}{2}, 4, 1, 2\}$

Q.) Find the auto correlation of the given sequence

$x(n) = \{5, \underset{\uparrow}{1}, 2, 1, 3\}$

III → Z-transform and its Application to the Analysis of LTI System

→ The z-transform plays the same role in the analysis of discrete-time signals and LTI systems as the Laplace transform does in the analysis of continuous-time signals and LTI systems.

→ In the z-domain (complex z-plane) the convolution of two time-domain signals is equivalent to multiplication of their corresponding z-transforms.

The Direct z-Transform

The z-transform of a discrete time signal $x(n)$ is defined as the power series:

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$, where z is a complex variable.

* The above relation is sometimes called the direct z-transform because it transforms the time-domain signal $x(n)$ into its complex-plane representation $X(z)$.
* The inverse procedure [i.e., obtaining $x(n)$ from $X(z)$] is called inverse z-transform.

→ For convinience, the z-transform of a signal $x(n)$ is denoted by

$$X(z) \equiv Z\{x(n)\}$$

→ Since the z-transform is an infinite power series, it exists only for those values of z for which this series converges.

* The region of convergence (ROC) of $X(z)$ is the

Set of all values of z for which X(z) attains a finite value. Thus any time we write a z-transform we also indicate its ROC.

Q.) Determine the z-transforms of the following finite duration signals:

1.) $x(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$

Ans- $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$X(z) = 1.z^{-0} + 2.z^{-1} + 5.z^{-2} + 7.z^{-3} + 0.z^{-4} + 1.z^{-5}$$

$$\Rightarrow X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

ROC: Entire z-plane except z=0.

2.) $x(n) = \{\underset{\uparrow}{0}, 0, 1, 2, 5, 7, 0, 1\}$ [H.W]

3.) $x(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

Ans- $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= 1.z^{2} + 2.z^{1} + 5.z^{0} + 7.z^{-1} + 0.z^{-2} + 1.z^{-3}$$

$$= z^{2} + 2z + 5 + 7z^{-1} + z^{-3}$$

ROC : Entire z-plane except z=0 and z=∞.

4.) $x(n) = \{1, 2, 3, \underset{\uparrow}{5}, 4, 0, 1\}$ [HW]

5.) $x(n) = \delta(n)$

Ans- $\delta(n) \xleftrightarrow{z} 1$

$\therefore X(z) = 1.$

ROC: Entire z-plane.

6) $x(n) = \delta(n-K)$, $K>0$.

Ans- $\delta(n-K) \xleftrightarrow{Z} z^{-K}$, $K>0$.

$\therefore X(z) = z^{-K}$

ROC: Entire z-plane except $z=0$.

7) $x(n) = \delta(n+K)$, $K>0$. [HW]

→ From the above examples, it is seen that the ROC of a finite-duration signal is the entire z-plane, except possibly the points $z=0$ and/or $z=\infty$.

* These points are excluded because $z^{K} (K>0)$ becomes unbounded for $z=\infty$ and $z^{-K}(K>0)$ becomes unbounded for $z=0$.

Z-Transform of infinite duration sequences

→ causal sequence

Q.) $x(n) = a^n u(n)$ → causal sequence

Ans- $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-1}$

$= \sum_{n=0}^{\infty} a^n z^{-n}$

$= \sum_{n=0}^{\infty} (az^{-1})^n$

$= \frac{1}{1-az^{-1}}$, for $|x|<1$

$= \frac{z}{z-a}$

Put $az^{-1} = x$.

For an infinite power series:

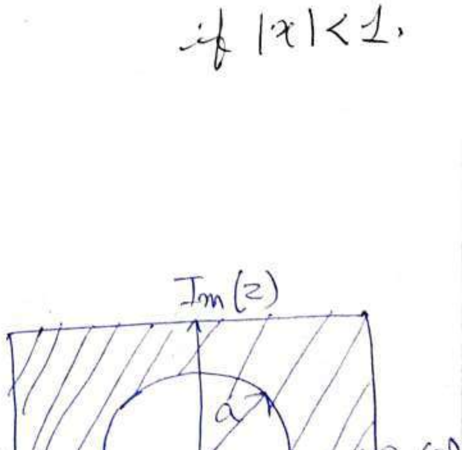
$x^n u(n) = \frac{1}{1-x}$

if $|x|<1$.

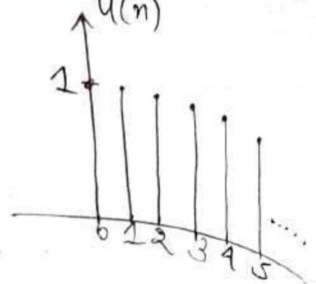
$\therefore$ ROC: $|x|<1$

$|az^{-1}| < 1$

$\Rightarrow a < z$ or $|z|>|a|$



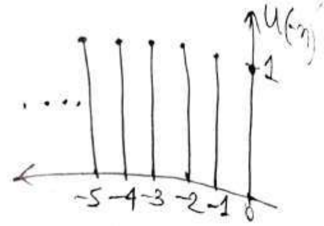
Q.) $x(n) = -b^n u(-n-1) \rightarrow$ non-causal / anti-causal sequence.



Ans $X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}$

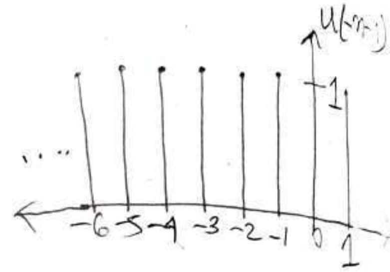
$= \sum_{-\infty}^{\infty} -b^n u(-n-1) z^{-n}$

$= \sum_{-\infty}^{+1} -b^n z^{-n}$



$= -\sum_{-\infty}^{+1} (b^{-1} z)^{-n}$

$\left[\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \right]$



$= -\sum_{n=1}^{\infty} \left((b^{-1} z)^{-n} \right)^{-1}$

$= -\sum_{n=1}^{\infty} (b^{-1} z)^n$

$= -\left[\sum_{n=0}^{\infty} (b^{-1} z)^n - 1 \right]$

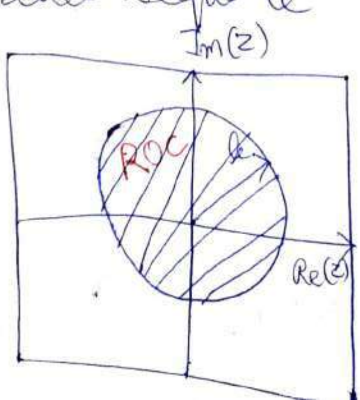
$= -\left[\frac{1}{1 - b^{-1} z} - 1 \right]$

$= \frac{-1}{1 - b^{-1} z} + 1 = \frac{\cancel{-1} + \cancel{1} - b^{-1} z}{1 - b^{-1} z}$

$= \frac{-z/b}{1 - z/b} = \frac{-z}{b - z} = \frac{z}{z-b}$

$\therefore$ ROC of this sequence is $|b^{-1} z| < 1$.

$|z| < |b|$



Q) $x(n) = a^n u(n) - b^n u(-n-1)$

Ans- $X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$

$$= \sum_{n=-\infty}^{\infty} \{a^n u(n) - b^n u(-n-1)\} z^{-n}$$

$$\cancel{= \sum_{n=-\infty}^{\infty} \{a^n u(n) + (-}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{n=1}^{\infty} (b^{-1} z)^n$$

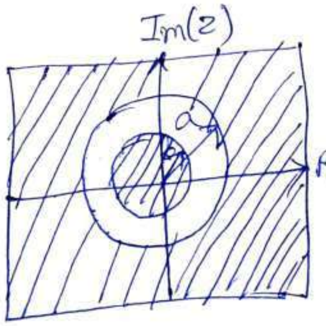
$$= \sum_{n=0}^{\infty} (a z^{-1})^n - \sum_{n=1}^{\infty} (b^{-1} z)^n$$

$$= \frac{z}{z-a} + \frac{z}{z-b}$$

$\frac{z}{z-a} \downarrow$ ROC: $|z| > |a|$

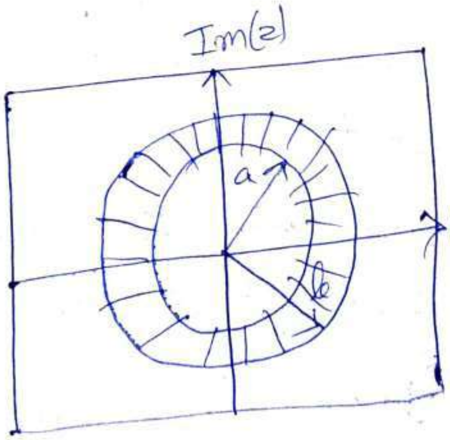
$\frac{z}{z-b} \downarrow$ ROC: $|z| < |b|$

→ If $|b| < |a|$



ROC $= |b| > |z| > |a|$

→ If $|a| < |b|$



ROC $= |a| < |z| < |b|$

The Inverse z-Transform

The process of transforming from z-domain to time domain is called inverse z-transform.

Suppose we multiply z^{n-1} to both sides and integrate over a closed contour C within the ROC of $X(z)$ we get;

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\oint_C X(z) z^{n-1} dz = \oint_C \sum_{K=-\infty}^{\infty} x(K) z^{n-1-K} dz.$$

So, the desired inversion formula is:

$$\boxed{x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz}$$

properties of ROC

→ ROC is a ring or disc in the z-plane centred at the origin.

→ ROC must be a connected region

→ ROC does not contain any pole.

→ If $x(n)$ is a causal sequence, ROC is all the values of z except $z=0$.

→ If $x(n)$ is non-causal sequence, then ROC is all the values of z except $z=\infty$.

→ ROC of any Linear time invariant signal contains unit circle.

Properties of Z-transform :-

1) Linearity

If $X(z) = Z[x(n)]$

then, $Z[a_1 x_1(n) + a_2 x_2(n)] = a_1 X_1(z) + a_2 X_2(z)$

$X_1(z) = Z[x_1(n)]$

$X_2(z) = Z[x_2(n)]$

Proof :- $Z[a_1 x_1(n) + a_2 x_2(n)]$

$$= \sum_{-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] z^{-n}$$

$$= \sum_{-\infty}^{\infty} a_1 x_1(n) z^{-n} + \sum_{-\infty}^{\infty} a_2 x_2(n) z^{-n}$$

$$= a_1 X_1(z) + a_2 X_2(z).$$

2) Time Shift

if $X(z) = Z[x(n)]$

then, $Z[x(n-m)] = z^{-m} X(z)$

Proof :- $Z[x(n-m)]$

let $l = n-m$

$\Rightarrow n = l+m$

$$= \sum_{-\infty}^{\infty} x(n-m) z^{-n}$$

$$= \sum_{l=-\infty}^{\infty} x(l) z^{-(l+m)}$$

$$= \sum_{l=-\infty}^{\infty} z^{-m} x(l) . z^{-l}$$

$$= z^{-m} \sum_{l=-\infty}^{\infty} x(l) . z^{-l}$$

$$= z^{-m} X(z)$$

3.) Time Reversal

If $X(z) = z[x(n)]$

then, $z[x(-n)] = X(z^{-1})$

Proof

$$z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$ let $-n = l$

$$= \sum_{l=-\infty}^{\infty} x(l) z^{l}$$

$$= \sum_{l=-\infty}^{\infty} x(l)(z^{-1})^{-l}$$

$$= X(z^{-1})$$

4.) Scaling in z-domain

$$x(n) \xleftrightarrow{z} X(z) \qquad ROC: r_1 < |z| < r_2$$

then $a^n x(n) \longleftrightarrow X(a^{-1}z)$ ROC: $|a|r_1 < |z| < |a|r_2$

for any constant real or complex value of a.

Proof: $$z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1}z)^{-n}$$

$$= X(a^{-1}z)$$

Q.) Determine the ROC and z-transform of the signal

$$x(n) = [3(2^n) - 4(3^n)]u(n)$$

Ans If we define the signals as

$$x_1(n) = 2^n u(n)$$

$$x_2(n) = 3^n u(n)$$ then $x(n) = 3x_1(n) - 4x_2(n)$

According to linearity property,

$$X(z) = 3X_1(z) - 4X_2(z)$$

we know, $a^n u(n) \xleftrightarrow{z} \dfrac{1}{1-az^{-1}}$, ROC: $|z| > |a|$

$x_1(n) = 2^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-2z^{-1}} = X_1(z)$, ROC : $|z| > 2$

$x_2(n) = 3^n u(n) \longleftrightarrow X_2(z) = \frac{1}{1-3z^{-1}}$, ROC : $|z| > 3$

Thus, overall z-transform is

$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}, \quad \text{ROC} : |z| > 3$$

Q.) Determine the z-transform of the signal :

a) $x(n) = (\cos \omega_0 n) u(n)$

Ans - We know from Euler's formula,

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\therefore x(n) = (\cos \omega_0 n) u(n)$$
$$= \frac{1}{2} e^{j\omega_0 n} u(n) + \frac{1}{2} e^{-j\omega_0 n} u(n)$$

$$\Rightarrow X(z) = \frac{1}{2} z\left[e^{j\omega_0 n} u(n)\right] + \frac{1}{2} z\left[e^{-j\omega_0 n} u(n)\right]$$

So, we know,

$$e^{j\omega_0 n} \cdot u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-e^{j\omega_0} z^{-1}}, \quad \text{ROC} : |z| > 1$$

$$e^{-j\omega_0 n} \cdot u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-e^{-j\omega_0} z^{-1}}, \quad \text{ROC} : |z| > 1$$

Thus, $X(z) = \frac{1}{2} \frac{1}{1-e^{j\omega_0} z^{-1}} + \frac{1}{2} \frac{1}{1-e^{-j\omega_0} z^{-1}}$

After some simple algebraic manipulation, we get :

$$(\cos \omega_0 n) u(n) \overset{z}{\longleftrightarrow} \frac{1 - z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0 + z^{-2}}, \quad \text{ROC} : |z| > 1.$$

b) $x(n) = (\sin \omega_0 n) u(n)$.

Ans - From Euler's formula,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$\therefore x(n) = (\sin\omega_0 n)u(n) = \frac{1}{2j}\left[e^{j\omega_0 n}u(n) - e^{-j\omega_0 n}u(n)\right]$

$\Rightarrow X(z) = \frac{1}{2j}\left(\frac{1}{1-e^{j\omega_0}z^{-1}} - \frac{1}{1-e^{-j\omega_0}z^{-1}}\right)$, ROC: $|z|>1$

So, finally we get:

$$(\sin\omega_0 n)u(n) \xleftrightarrow{z} \frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}, \text{ ROC}: |z|>1$$

5.) Differentiation in z-domain

$$x(n) \xleftrightarrow{z} X(z)$$

$$n\,x(n) \xleftrightarrow{z} -z\frac{dX(z)}{dz}$$

Proof ~~x(n)~~ $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$

Differentiating both sides of the above eqn. we get

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n).(-n)z^{-n-1}$$

$$= -z^{-1}\sum_{n=-\infty}^{\infty}[n\,x(n)]z^{-n}$$

$$= -z^{-1}Z[n\,x(n)]$$

6.) Convolution of two Sequences

If $x_1(n) \xleftrightarrow{z} X_1(z)$

$x_2(n) \xleftrightarrow{z} X_2(z)$

Then $x(n) = x_1(n) * x_2(n) \xleftrightarrow{z} X(z) = X_1(z).X_2(z)$

Q.) Compute the convolution $x(n)$ of the signal.

$$x_1(n) = \{1, -2, 1\}$$

$$x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases}$$

Ans- $x_1(n) = \{1, -2, 1\}$

$x_2(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 1, 1\}$

$x(n) = x_1(n) * x_2(n)$

$X_1(z) = 1 - 2z^{-1} + z^{-2}$

$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$

According to convolution property of z-transform:

$$X(z) = X_1(z) \cdot X_2(z)$$
$$= (1 - 2z^{-1} + z^{-2}) \cdot (1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5})$$
$$= 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} - 2z^{-1} - 2z^{-2} - 2z^{-3} - 2z^{-4} - 2z^{-5} - 2z^{-6} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} + z^{-7}$$
$$= 1 - z^{-1} - z^{-6} + z^{-7}$$

$\therefore x(n) = \{\underset{\uparrow}{1}, -1, 0, 0, 0, 0, -1, 1\}$

7) Correlation of two Sequences

If $x_1(n) \xleftrightarrow{Z} X_1(z)$

$x_2(n) \xleftrightarrow{Z} X_2(z)$

then $r_{x_1 x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2(n-l) \xleftrightarrow{Z}$

$$R_{x_1 x_2}(z) = X_1(z) \cdot X_2(z^{-1})$$

or $\boxed{r_{x_1 x_2}(l) = x_1(l) * x_2(-l)}$ → Auto Correlation

$\boxed{r_{xy}(l) = x(l) * y(-l)}$ → Cross-Correlation

Q.) Determine the auto-correlation sequence of the Signal: $x(n) = a^n u(n)$, $-1 < a < 1$.

Ans- Auto-correlation of two sequences is

$$R_{xx}(z) = Z\{r_{xx}(l)\} = X(z) \cdot X(z^{-1})$$
$$= x(n) * x(-n)$$

$$X(z) = \frac{1}{1-az^{-1}} \quad ROC: |z| > |a| \quad \text{(causal signal)}$$

$$x(-n) \overset{z}{\longleftrightarrow} X(z^{-1}) = \frac{1}{1-az} \quad ROC: |z| < \frac{1}{|a|} \quad \text{(anti-causal signal)}$$

$$\therefore R_{xx}(z) = \frac{1}{1-az^{-1}} \cdot \frac{1}{1-az}$$

$$= \frac{1}{1-a(z+z^{-1})+a^2}, \quad ROC: |a| < |z| < \left|\frac{1}{a}\right|$$

8.) Multiplication of two sequences

If $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$

$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$

$$x(n) = x_1(n) \cdot x_2(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{2\pi j} \oint_C X_1(v) \cdot X_2\left(\frac{z}{v}\right) v^{-1} dv$$

9.) Parseval's Theorem

If $x_1(n)$ and $x_2(n)$ are complex valued sequences

$$\boxed{\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi j} \oint X_1(v) \cdot X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv}$$

10.) Initial Value Theorem

If $x(n)$ is causal [i.e., $x(n) = 0$, for $n < 0$], then

$$\boxed{x(0) = \lim_{z \to \infty} X(z)}$$

11.) Final Value Theorem

$$\boxed{x(\infty) = \lim_{z \to 1} (z-1) X_T(z)}$$

Rational Z-Transform

An important family of Z-Transform are those for which $X(z)$ is a rational function, that is a ratio of two polynomials in z^{-1} (or z).

Poles and Zeros

The zeros of a Z-Transform $X(z)$ are the values of z for which $X(z)=0$. The poles of a Z-Transform are the values of z for which $X(z)=\infty$.

If $X(z)$ is a rational function, then

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

If $a_0 \neq 0$ and $b_0 \neq 0$, we can avoid the negative power of z by factoring out the terms $b_0 z^{-M}$ and $a_0 z^{-M}$.

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0}{a_0} z^{-M+N} \cdot \frac{(z-z_1)(z-z_2)(z- \cdots z_M)}{(z-p_1)(z-p_2)(z-p_3)\cdots(z-p_N)}$$

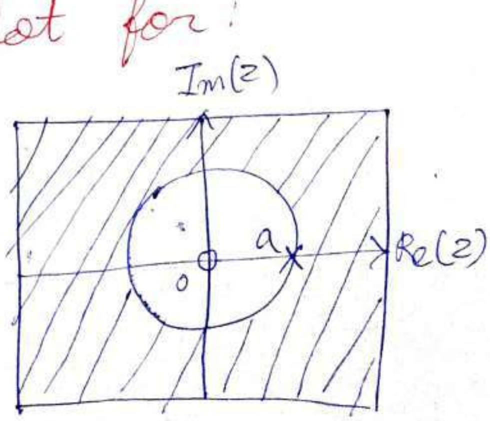
* Graphically we represent pole as (X) and zeros as (o).

Q.) Determine the pole-zero plot for:

$$x(n) = a^n u(n),\ a>0.$$

Ans- $X(z) = \dfrac{1}{1-a z^{-1}} = \dfrac{z}{z-a}$, ROC: $|z| > a$.

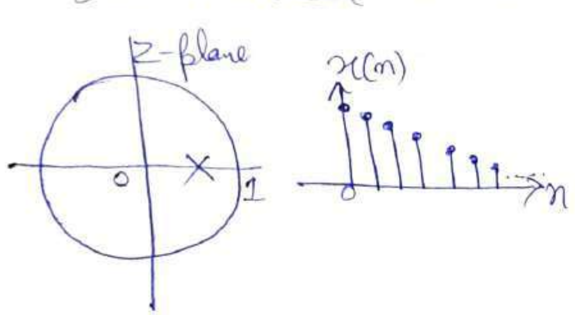
$X(z)$ has one zero at $z_1 = 0$ and one pole at $p_1 = a$

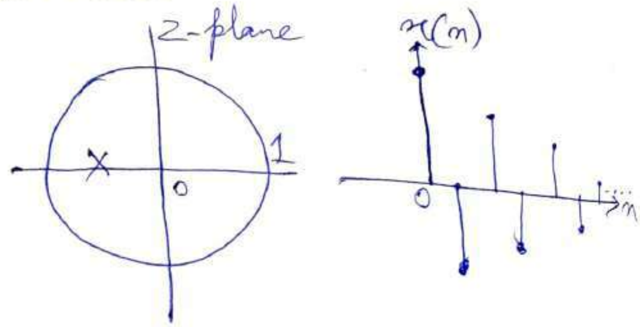


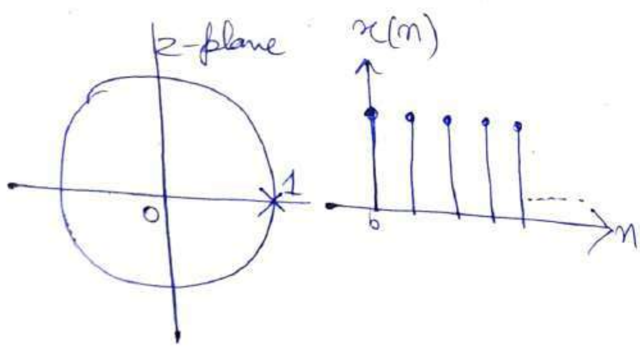
Pole Location and Time Domain Behaviour for Causal Signals

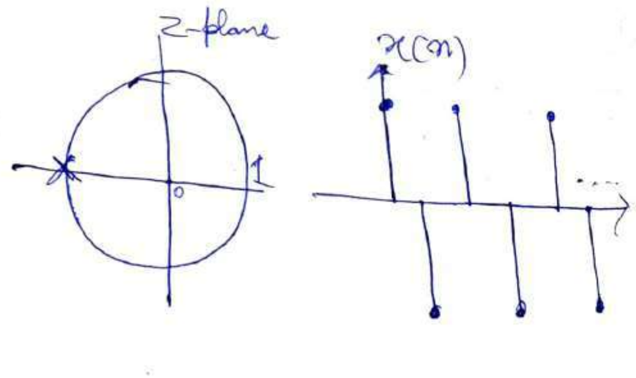
Here, we will consider the relation between z-plane location of a pole pair and the form (shape) of the corresponding signal in the time domain.

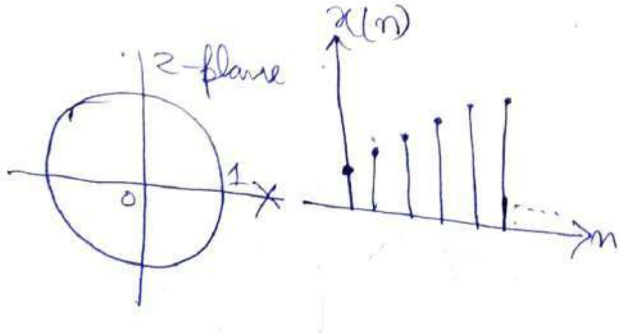
→ We deal exclusively with real, causal. We see that the characteristics behaviour of causal signal depends on whether the poles of the transform are contained in the region $|z|<1$, or in the region $|z|>1$, or on the circle $|z|=1$. Since the circle $|z|=1$ has a radius of 1, it is called the unit circle

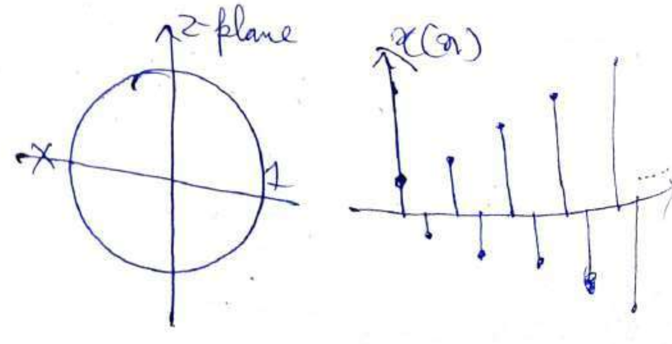












The System Function of a Linear Time-Invariant System

The input sequence $x(n)$ can be obtained by computing the convolution of $x(n)$ with the unit sample response of the system it expresses in z-domain.

$$Y(z) = H(z) \cdot X(z)$$

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$
$$y(n) \overset{z}{\longleftrightarrow} Y(z)$$
$$h(n) \overset{z}{\longleftrightarrow} H(z)$$

$$\Rightarrow \boxed{H(z) = \frac{Y(z)}{X(z)}}$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

→ A system function is described by linear time constant co-efficient differential equation.

$$y(n) = -a_K \sum_{K=1}^{N} y(n-K) + b_K \sum_{K=0}^{M} x(n-K)$$

$$y(z) = -a_K \sum_{K=1}^{N} z^{-K} y(z) + b_K \sum_{K=0}^{M} z^{-K} x(z)$$

$$Y(z)\left[1 + a_K \sum_{K=1}^{N} z^{-K}\right] = b_K \, X(z) \sum_{K=0}^{M} z^{-K} X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{b_K \sum_{K=0}^{M} z^{-K} X(z)}{1 + a_K \sum_{K=1}^{N} z^{-K}}$$

Q) Determine the system function of the unit sample response of the system described by the differential equation: $y(n) = \frac{1}{2} y(n-1) + 2x(n)$

Ans By computing the z-transform of the differential equation:

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z)$$

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = 2X(z)$$

$$\Rightarrow Y(z)\left(1 - \frac{1}{2} z^{-1}\right) = 2X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{1}{2} z^{-1}}$$

$$H(z) = \frac{2}{1 - \frac{1}{2} z^{-1}}$$

$$h(n) = 2\left(\frac{1}{2}\right)^n . u(n)$$

Discuss Inverse Z-Transform

The inverse Z-Transform is given by:

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) . z^{n-1} dz \quad \text{——— (1)}$$

where the integral is a contour integral on a closed path C that encloses the origin a lies within the region of convergence of X(z).

→ Various methods for evaluation of inverse Z-Transform are:

1) Direct evaluation of eqn. (1) by contour integrati
2) Partial fraction expansion and table look up.
3) Power series expansion.

Partial - Fraction Expansion

In the table look up method, the function $X(z)$ is expressed as a linear combination:

$$X(z) = a_1 X_1(z) + a_2 X_2(z) + \ldots\ldots + a_K X_K(z)$$

Inverse of $X(z)$ is $x(n)$

$$\therefore x(n) = a_1 x_1(n) + a_2 x_2(n) + \ldots\ldots + a_K x_K(n)$$

Without loss of generality, $a_0 = 1$

$$X(z) = \frac{B(z)}{A(z)}$$

$$= \frac{b_0 + b_1 z^{-1} + \ldots\ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots\ldots\ldots + a_N z^{-N}}$$

→ A rational function is called proper if $a_N \neq 0$ and $M < N$.

To simplify our discussion, we eliminate negative powers of z by multiplying both the numerator and denominator by z^N:

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \ldots\ldots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \ldots\ldots + a_N}$$

which contains only positive power of z.

Since $N > M$, the function

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \ldots\ldots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \ldots\ldots + a_N}$$

is also always proper.

Distinct Poles

Suppose that the poles $p_1, p_2, \ldots p_N$ are all different (distinct).

We get an expansion of the form:

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \cdots + \frac{A_N}{z-p_N}$$

Q.) 1.) Determine the partial fraction expansion of the proper function; and inverse z-Transform:

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

Ans- We eliminate the negative power by multiplying z^2 with both numerator and denominator.

$$X(z) = \frac{z^2}{z^2-1.5z+0.5}$$

The poles of $X(z)$ are $p_1 = 1$ and $p_2 = 0.5$.

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

* A simple method to determine A_1 and A_2 is to multiply the equation by the denominator term $(z-1)(z-0.5)$.

Thus, we obtain

$$z = (z-0.5)A_1 + (z-1)A_2$$

$$\boxed{A_K = (z-p_K) \times \frac{X(z)}{z}\bigg|_{z=p_K}}$$

$$A_1 = (z-p_1) \cdot \frac{X(z)}{z}\Big|_{z=p_1}$$

$$= \cancel{(z-1)} \times \frac{z}{\cancel{(z-1)}(z-0.5)}\Big|_{z=1}$$

$$= \frac{z}{z-0.5}\Big|_{z=1}$$

$$= \frac{1}{1-0.5} = \frac{1}{1/2} = \ldots 2$$

$$A_2 = (z-p_2) \cdot \frac{X(z)}{z}\Big|_{z=p_2}$$

$$= \cancel{(z-0.5)} \cdot \frac{z}{(z-1)\cancel{(z-0.5)}}\Big|_{z=0.5}$$

$$= \frac{0.5}{(0.5-1)} = -1.$$

$\therefore$ The result of the partial fraction expansion is

$$\frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

$$\therefore X(z) = \frac{2z}{z-1} - \frac{z}{z-0.5}$$

$$\boxed{x(n) = 2(1)^n u(n) - \left(\tfrac{1}{2}\right)^n u(n)}$$

Q.(2) Determine the partial fraction expansion of

$$X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$$

Ans- To eliminate negative power of z, we multiply both numerator and denominator by z^2.

$$\frac{X(z)}{z} = \frac{z+1}{z^2-z+0.5}$$

The poles of $X(z)$ are complex conjugates.

$$p_1 = \frac{1}{2} + j\frac{1}{2}$$

$$p_2 = \frac{1}{2} - j\frac{1}{2}$$

We seek an expression:

$$\frac{X(z)}{z} = \frac{z+1}{(z-p_1)(z-p_2)} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2}$$

$$A_1 = (z-p_1) \cdot \left.\frac{X(z)}{z}\right|_{z=p_1}$$

$$= \cancel{(z-p_1)} \times \left.\frac{z+1}{\cancel{(z-p_1)}(z-p_2)}\right|_{z=p_1}$$

$$= \frac{\frac{1}{2} + j\frac{1}{2} + 1}{\frac{1}{2} + j\frac{1}{2} - \frac{1}{2} + j\frac{1}{2}} = \frac{1}{2} - j\frac{3}{2}$$

$$A_2 = (z-p_2) \cdot \left.\frac{X(z)}{z}\right|_{z=p_2}$$

$$= \cancel{(z-p_2)} \cdot \left.\frac{z+1}{(z-p_1)\cancel{(z-p_2)}}\right|_{z=p_2}$$

$$= \frac{\frac{1}{2} - j\frac{1}{2} + 1}{\frac{1}{2} - j\frac{1}{2} - \frac{1}{2} - j\frac{1}{2}} = \frac{1}{2} + j\frac{3}{2}$$

Multi - order Poles

Q.3) Determine the partial fraction of:

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}\text{, and the inverse z-Transform.}$$

Ans- We express the given equation in positive powers of z.

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_1}{(z+1)} + \frac{A_2}{(z-1)} + \frac{A_3}{(z-1)^2}$$

$X(z)$ has a simple pole at $p_1 = -1$ and double pole at $p_2 = p_3 = 1$.

$$A_1 = \cancel{(z+1)} \cdot \frac{z^2}{\cancel{(z+1)}(z-1)^2}\Big|_{z=-1}$$

$$= \frac{(-1)^2}{(-1-1)^2} = \frac{1}{4}$$

$$A_2 = \frac{d}{dz}\left[\cancel{(z-1)^2} \cdot \frac{z^2}{(z+1)\cancel{(z-1)^2}}\Big|_{z=1}\right]$$

$$= \frac{d}{dz}\left[\frac{z^2}{z+1}\Big|_{z=1}\right]$$

$$= \frac{2z(z+1) - z^2}{(z+1)^2}\Big|_{z=1}$$

$$= \frac{2z^2 + 2z - z^2}{(z+1)^2}\Big|_{z=1}$$

$$= \frac{2+2-1}{4} = \frac{3}{4}$$

$$A_3 = (z-1)^2 \cdot \frac{z^2}{(z+1)(z-1)^2}\Big|_{z=1} = \frac{1}{2}$$

$$\therefore \frac{X(z)}{z} = \frac{A_1}{z+1} + \frac{A_2}{(z-1)} + \frac{A_3}{(z-1)^2}$$

$$= \frac{1/4}{z+1} + \frac{3/4}{z-1} + \frac{1/2}{(z-1)^2}$$

$$\Rightarrow X(z) = \frac{1/4\,z}{z+1} + \frac{3/4\,z}{z-1} + \frac{\frac{1}{2}z}{(z-1)^2}$$

$$\therefore \boxed{x(n) = \frac{1}{4}(-1)^n u(n) + \frac{3}{4}(1)^n u(n) + \frac{1}{2} n\, u(n)}$$

Q.) Determine the inverse Z-Transform of: $X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$ if ROC: a) $|z| > 1$, b) $|z| < 0.5$, c) $0.5 < |z| < 1$

Ans- $X(z) = \frac{2}{1 - z^{-1}} - \frac{1}{1 - 0.5z^{-1}}$

a) When ROC is $|z| > 1$, the signal $x(n)$ is causal.

$\therefore x(n) = 2(1)^n u(n) - (0.5)^n u(n) = (2 - 0.5)^n u(n)$.

b) When ROC is $|z| < 0.5$, the signal $x(n)$ is anti-causal.

$\therefore x(n) = [-2 + 0.5]^n u(-n-1)$

c) When ROC is $0.5 < |z| < 1$ is a ring, which implies the signal $x(n)$ is two-sided. Thus, one of the terms corresponds to a ~~usal~~ causal signal and other to anti-causal signal.

$|z| > 0.5$ and $|z| < 1 \Rightarrow p_2 = 0.5$ is causal part and $p_1 = 1$ is the anti-causal part.

$\therefore x(n) = -2(1)^n u(-n-1) - (0.5)^n u(n)$.

Unit-IV

Discuss Fourier Transform: Its Applications and Properties

To perform frequency analysis on a discrete-time signal $x(n)$, we convert the time-domain sequence to an equivalent frequency-domain representation is given by $X(\omega)$.

$\boxed{x(n) \xleftrightarrow{DTFT} X(e^{j\omega})}$ $\boxed{x(n) \xleftrightarrow{DFT} X(\omega)}$ $\boxed{x(t) \xleftrightarrow{CTFT} X(j\omega)}$

Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

→ Aperiodic finite-energy signals have continuous spectra

Let us consider an aperiodic discrete-time signal $x(n)$ having Fourier Transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Suppose that we sample $X(\omega)$ periodically in frequency at a spacing of $\delta\omega$ radians between successive samples.

$X(\omega)$ is periodic with period 2π.

We take N-equidistant samples in the interval $0 \leq \omega \leq 2\pi$ with spacing $\delta\omega = 2\pi/N$.

$$\omega = 2\pi K/N$$

$$\therefore X\left(\frac{2\pi}{N}K\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi Kn/N}, \quad K = 0, 1, \ldots, N-1.$$

$$= \cdots + \sum_{n=-N}^{-1} x(n)\, e^{-j2\pi Kn/N} + \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi Kn/N}$$

$$+ \sum^{2N-1} x(n)\, e^{-j2\pi Kn/N} \ldots$$

The above eqn. can be expanded into a Fourier Series as

$$x_p(n) = \sum_{K=0}^{N-1} C_K e^{j2\pi Kn/N}, \quad n = 0, 1, \ldots, N-1$$

Fourier series coefficient

$$\boxed{C_K = \frac{1}{N} \sum_{n=0}^{N-1} x_p e^{-j2\pi Kn/N}, \quad K = 0, 1, \ldots, N-1}$$

Example - Consider the Signal $x(n) = a^n u(n)$
Determine DFT.

Ans -
$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{\infty} a^n \cdot e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{\infty} (a \cdot e^{-j\omega})^n$$
$$= \frac{1}{1 - a e^{-j\omega}}$$

The Discrete Fourier Transform (DFT)

The frequency samples $X(2\pi K/N)$, $K = 0, 1, \ldots, N-1$ uniquely represent the finite duration sequence $x(n)$. The 'L' equidistant samples of $X(\omega)$ are sufficient to reconstruct $X(\omega)$. Padding the sequence $x(n)$ with $N-L$ zeros and computing an N-point DFT results in a 'better display' of Fourier Transform $X(\omega)$.

→ For a finite duration sequence $x(n)$ of length L [i.e, $x(n)=0$ for $n<0$ and $n\geq L$], the Fourier Transform is given by:

$$\boxed{X(\omega)=\sum_{n=0}^{L-1} x(n)\, e^{-j\omega n}, \quad 0\leq\omega\leq 2\pi}$$

* When we sample $X(\omega)$ at equally spaced frequencies $\omega_K = 2\pi K/N$, $K=0,1,2,\ldots,N-1$, where $N>L$, the resultant samples are:

$$\boxed{X(K)=\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi Kn/N}, \quad K=0,1,2,\ldots,N-1.}$$

The above equation is called the Discrete Fourier Transform (DFT) of $x(n)$.

$$\boxed{x(n)=\frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{j2\pi Kn/N}, \quad n=0,1,2,\ldots,N-1}$$

is called the inverse DFT (IDFT).

The DFT as a Linear Transform

$$X(K)=\sum_{n=0}^{N-1} x(n)\, W_N^{Kn}, \quad K=0,1,\ldots,N-1$$

$$x(n)=\frac{1}{N}\sum_{K=0}^{N-1} X(K)\, W_N^{-Kn}, \quad n=0,1,\ldots,N-1$$

The above equations represents the DFT and IDFT.

where, by definition, $\boxed{W_N = e^{-j2\pi/N}}$ and $\boxed{W_N^K = e^{-j2\pi K/N}}$

which is an N^{th} root of unity

It is ~~useful~~ instructive to view the DFT and IDFT as linear transformations on sequences $x(n)$ and $X(K)$. Let us define an N-point vector x_N of the signal sequence $x(n), n=0,1,\ldots N-1$, an N-point vector X_N of frequency samples and an $N \times N$ matrix W_N as

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \quad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & & & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

With these definitions the N-point DFT may be expressed in matrix form as:

$$\boxed{X_N = W_N x_N} \quad \text{———— DFT}$$

IDFT

$$x_N = W_N^{-1} X_N$$

$$\boxed{x_N = \frac{1}{N} W_N^{*} X_N} \quad \text{———— IDFT}$$

where W_N^* denotes the complex conjugate of the matrix W_N.

By comparing the expressions of DFT and IDFT we conclude that

$$W_N^{-1} = \frac{1}{N} W_N^*$$

which in turn implies that

$$W_N W_N^* = N I_N$$

where I_N is an $N \times N$ identity matrix. Therefore the matrix W_N in the transformation is an orthogonal (Unitary) matrix.

Q.) Compute the DFT of the four-point sequence : $x(n) = (0 \quad 1 \quad 2 \quad 3)$.

Ans- The 1st step is to determine the matrix W_4.

$$W_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

We know, $W_N^K = e^{-j\frac{2\pi K}{N}}$

$$\Rightarrow W_4^0 = e^{-j\frac{2\pi 0}{4}} = e^0 = 1$$

$$W_4^1 = e^{-j\frac{2\pi \times 1}{N}} = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$$
$$= 0 - j(1)$$
$$= -j$$

$$W_4^2 = e^{-j\frac{2\pi \times 2}{4}} = e^{-j\pi} = \cos\pi - j\sin\pi$$
$$= -1 - 0 = -1$$

$$W_4^3 = e^{-j\frac{\cancel{2}\pi \times 3}{\cancel{4}2}} = e^{-j\frac{3\pi}{2}} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$$
$$= j$$

$$W_4^4 = e^{-j\frac{2\pi \times \cancel{4}}{\cancel{4}}} = e^{-j2\pi} = \cos 2\pi - j\sin 2\pi$$
$$= 1$$

$$W_4^6 = e^{-j\frac{\cancel{2}\pi \times \cancel{6}3}{\cancel{4}\cancel{2}}} = e^{-j3\pi} = \cos 3\pi - j\sin 3\pi$$
$$= -1$$

$$W_4^9 = e^{-j\frac{\cancel{2}\pi \times 9}{\cancel{4}2}} = e^{-j\frac{9\pi}{2}} = \cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}$$
$$= -j$$

$$\therefore W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$X_4 = W_4 \cdot x_4$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} (1\times0)+(1\times1)+(1\times2)+(1\times3) \\ (1\times0)+(-j\times1)+(-1\times2)+(j\times3) \\ (1\times0)+(-1\times1)+(1\times2)+(-1\times3) \\ (1\times0)+(j\times1)+(-1\times2)+(-j\times3) \end{bmatrix} = \begin{bmatrix} 6 \\ -j-2+3j \\ -1+2-3 \\ j-2-3j \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$X_4 = \{6, -2+2j, -2, -2-2j\}$$

Q.2) Find the 4-point DFT of the sequence
$x(n) = \{1, 2, 1, 1\}$

Q.3) Compute 4-point DFT of the sequence
$x(n) = \{0, 2, 4, 6\}$

Q.4) compute 4-point DFT of the sequence
$x(n) = \{5, 6, 2, 1\}$

Difference Between DFT and DTFT

DFT	DTFT
→ DFT stands for Discrete Fourier Transform.	→ DTFT stands for Discrete-time Fourier Transform.
→ $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}}$	→ $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$
→ The range of a DFT sequence is finite. The range is from 0 to N-1. A DFT sequence contains only positive frequencies.	→ The range of a DTFT sequence is infinite. It starts from negative infinity ($-\infty$) to positive infinity (∞) and contains both positive and negative frequencies.
→ Non-continuous sequence.	→ Continuous Sequence.
→ A DFT sequence has periodicity, hence called periodic sequence with period N.	→ A DTFT sequence contains periodicity, hence called periodic sequence with period 2π.
→ The DFT can be calculated in computers as well as in digital processors as it does not contain any continuous variable of frequency.	→ The calculation of DTFT on computers and digital signal processors is always a problem as the DTFT deals with infinite length signals.
→ Suitable for computer implementation.	→ Not suitable for computer implementation.
→ Applications - Used in image processing	→ Applications - Used in the analysis of samples of a continuous function.

Q.) Determine the 8-point DFT of the sequence:

$x(n) = \{1, 1, 1, 1, 1, 1, 0, 0\}$

$$W_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ 1 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ 1 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ 1 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ 1 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{36} \\ 1 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ 1 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix}$$

We know, $W_N^K = e^{-j\frac{2\pi K}{N}}$

$W_8^1 = e^{-j\frac{2\pi \times 1}{8}} = e^{-j\pi/4} = \cos\pi/4 - j\sin\pi/4 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_8^2 = e^{-j\frac{2\pi \times 2}{8}} = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = 0 - j = j$

$W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_8^4 = -1$

$W_8^5 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$W_8^6 = j$

$W_8^7 = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$

$W_8^8 = 1$

$W_8^9 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_8^{10} = -j$

$W_8^{12} = -1$

$W_8^{14} = +j$

$W_8^{15} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$W_8^{18} = -j$

$W_8^{21} = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$W_8^{20} = -1$

$W_8^{24} = 1$

$W_8^{25} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_8^{28} =$

$W_8^{30} = j$

$W_8^{35} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_8^{36} = -1$

$W_8^{42} = -j$

$W_8^{49} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$\underline{D}$ $X_8 = W_8 \cdot x_8$

$$\Rightarrow X_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt2}-\frac{j}{\sqrt2} & -j & -\frac{1}{\sqrt2}-\frac{j}{\sqrt2} & -1 & -\frac{1}{\sqrt2}+\frac{j}{\sqrt2} & j & \frac{1}{\sqrt2}+j/\sqrt2 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -1/\sqrt2 - j/\sqrt2 & j & \frac{1}{\sqrt2}-j/\sqrt2 & -1 & \frac{1}{\sqrt2}+j/\sqrt2 & -j & -\frac{1}{\sqrt2}+j/\sqrt2 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{\sqrt2}+j/\sqrt2 & j & \frac{1}{\sqrt2}+j/\sqrt2 & -1 & \frac{1}{\sqrt2}-j/\sqrt2 & j & -\frac{1}{\sqrt2}-\frac{j}{\sqrt2} \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & \frac{1}{\sqrt2}+j/\sqrt2 & j & -\frac{1}{\sqrt2}+j/\sqrt2 & -1 & -\frac{1}{\sqrt2}-j/\sqrt2 & -j & \frac{1}{\sqrt2}-j/\sqrt2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -\frac{1}{\sqrt2} - j\frac{1}{\sqrt2} - j \\ 1-j \\ \frac{1}{\sqrt2} + j - \frac{j}{\sqrt2} \\ \frac{1}{\sqrt2} - j + j/\sqrt2 \\ 1+j \\ -\frac{1}{\sqrt2} + j + j/\sqrt2 \end{bmatrix}$$

Q.) Compute 4-point IDFT of the sequence:
$y(k) = \{1, 0, 1, 0\}$

Ans- $Y(K)$ or $X(K) = \{1, 0, 1, 0\}$

$IDFT = \frac{1}{N} W_N^* X_N.$

$$x_4 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} (1\times1)+(1\times0)+(1\times1)+(1\times0) \\ (1\times1)+(j\times0)+(-1\times1)+(-j\times0) \\ (1\times1)+(-1\times0)+(1\times1)+(-1\times0) \\ (1\times1)+(-j\times0)+(-1\times1)+(j\times0) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/4 \\ 0 \\ 2/4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}$$

Relationship of DFT to other Transforms

* Relationship to the Fourier Series coefficients of a periodic Sequence.

A periodic sequence $\{x_p(n)\}$ with fundamental period N can be represented in a Fourier Series:

$$x_p = \sum_{K=0}^{N-1} c_K e^{j2\pi nK/N}, \quad -\infty < n < \infty.$$

where the Fourier Series co-efficients are given by

$$C_K = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j2\pi nK/N}, \quad K = 0, 1, \ldots, N-1.$$

By comparing these two equations, the DFT of this sequence is simply: $\boxed{X(K) = NC_K}$

* Relationship to the Fourier Transform of an Aperiodic Sequence

If $x(n)$ is an aperiodic finite energy sequence with Fourier Transform $X(\omega)$, which is sampled at N equally spaced frequencies $\omega_K = 2\pi K/N$, $K = 0, 1, \ldots N-1$

$$X(K) = X(\omega)\Big|_{\omega = \frac{2\pi K}{N}} = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\frac{2\pi n K}{N}}, \quad K = 0, 1, \ldots, N-1$$

- are the DFT coefficients of the periodic sequence of a period N, given by:

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

- Thus $x_p(n)$ is determined by aliasing $\{x(n)\}$ over the interval $0 \leq n \leq N-1$. The finite duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

$x(n)$ is finite duration and length $l \leq N$.

$$x(n) = \hat{x}(n), \quad 0 \leq n \leq N-1.$$

* Relationship to Z Transform

Let us consider a sequence $x(n)$ having z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

with an ROC that includes the unit circle.

If $X(z)$ is sampled at N equally spaced points on the unit circle $z_K = e^{j2\pi K/N}$, $0, 1, 2, \ldots, N-1$, we obtain

$$X(K) \equiv X(z)\Big|_{z = e^{j\frac{2\pi K n}{N}}}, \quad K = 0, 1, \ldots, N-1$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi n K}{N}}$$

If the sequence $x(n)$ has finite duration of the length N or less, the sequence can be recovered from its N-point DFT. Hence its Z-transform is uniquely determined by its N-point DFT $X(K)$.

$$X(z) = \sum_{n=0}^{N-1} x(n) \cdot z^{-n}$$

$$= \sum_{K=0}^{N-1} \left[\frac{1}{N} \sum_{n=0}^{N-1} X(K) \cdot e^{j\frac{2\pi n K}{N}} \right] z^{-n}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) \sum_{n=0}^{N-1} \left(e^{-j\frac{2\pi K n}{N}} z^{-1} \right)^{n}$$

$$= \frac{1-Z^{-N}}{N} \sum_{K=0}^{N-1} \frac{X(K)}{1-e^{j2\pi Kn/N} z^{-1}}$$

When evaluated on the unit circle, the Fourier Transform of the finite duration sequence in terms of its DFT, in the form

$$\boxed{X(\omega) = \frac{1-e^{j\omega N}}{N} \sum_{K=0}^{N-1} \frac{X(K)}{1-e^{-j(\omega-2\pi K/N)}}}$$

* Relationship to the Fourier Series coefficients of a continuous-time Signal

Suppose that $x_a(t)$ is a continuous-time periodic signal with fundamental period $T_p = 1/F_0$. The signal can be expressed in a Fourier series

$$x_a(t) = \sum_{K=-\infty}^{\infty} c_K e^{j2\pi K t F_0}$$

where c_k are the Fourier coeffecients.
If we sample $x_a(t)$ at an uniform rate $F_s = N/T$ $= 1/T$, we obtain the discrete-time sequence

$$x(n) \equiv x_a(nT) \equiv \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 nT}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kn/N}$$

$$= \sum_{k=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} c_{k-lN} \right] e^{j2\pi kn/N}$$

It is clear that

$$X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N\tilde{c}_k$$

$$\boxed{\tilde{c}_k = \sum_{l=-\infty}^{\infty} c_{k-lN}}$$

Thus, the $\{\tilde{c}_k\}$ sequence is an aleased version the sequence $\{c_k\}$.

Properties of DFT

$$DFT = X(k) = \sum_{n=0}^{N-1} x(n) . W_N^{kn} \;, \quad k = 0, 1, \ldots, N-1$$

$$IDFT = \frac{1}{N} \sum_{k=0}^{N-1} X(k) . W_N^{-kn} \;, \quad n = 0, 1, \ldots, N-1$$

where $W_N = e^{-j2\pi/N}$

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

Periodicity -

If $x(n)$ and $X(K)$ are an N-point DFT pair then

$$x(n+N) = x(n) \text{ for all } n.$$
$$X(K+N) = X(K) \text{ for all } K.$$

These periodicities in $x(n)$ and $X(K)$ follow the immediately from formulas for DFT and IDFT.

Linearity -

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$

then for any real-valued or complex-valued constants a_1 and a_2

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(K) + a_2 X_2(K)$$

Circular Symmetries of a Sequence

The N-point DFT of a finite duration sequence $x(n)$, of length $L \leq N$, is equivalent to the N-point DFT of a periodic sequence $x_p(n)$ of period N, which is obtained by the periodically extending $x(n)$:

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

Now suppose that we shift the periodic sequence $x_p(n)$ by K units to the right.

$$x'_p(n) = x_p(n-K) = \sum_{l=-\infty}^{\infty} x(n-K-lN)$$

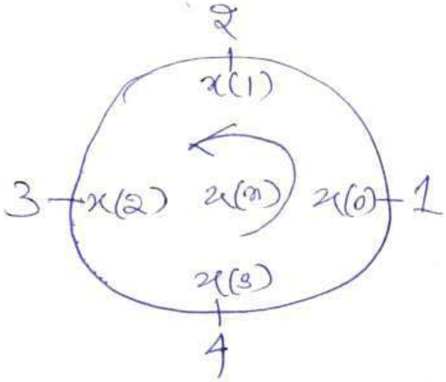
Q) A sequence is given as $x(n) = \{1, 2, 3, 4\}$ Circular shift the sequency by '2'.

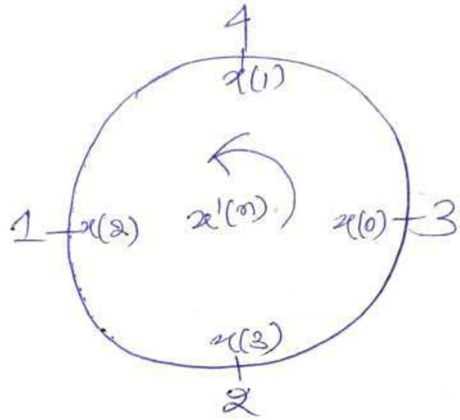
Ans- $K=2$
$N=4$

$$x'_p(n) = x_p(n-2)$$

* The finite duration sequence

$$x'(n) = \begin{cases} x'_p(n), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$





* An N-point sequence is called circularly even if it is symmetric about the point zero on the circle.

$$\Rightarrow x(N-n) = x(n), \quad 1 \leq n \leq N-1.$$

- A N-point sequence is called circularly odd if it is asymmetric about the point zero on the circle.

$$\Rightarrow x(N-n) = -x(n), \quad 1 \leq n \leq N-1.$$

→ For periodic sequences:

even: $x_p(n) = x_p(-n) = x_p(N-n)$

odd: $x_p(n) = -x_p(-n) = -x_p(N-n)$

- If the periodic sequence is complex valued, we have:

conjugate even: $x_p(n) = x_p^*(N-n)$

conjugate odd: $x_p(n) = -x_p^*(N-n)$

$$x_p = x_{p_e}(n) + x_{p_o}(n)$$

where $x_{p_e}(n) = \frac{1}{2}\left[x_p(n) + x_p^*(N-n)\right]$

$$x_{p_o}(n) = \frac{1}{2}\left[x_p(n) - x_p^*(N-n)\right]$$

Symmetric Properties of the DFT

$$x(n) = x_R(n) + j x_I(n), \quad 0 \leq n \leq N-1$$

$$X(K) = X_R(K) + j X_I(n), \quad 0 \leq K \leq N-1$$

Substituting into the expression of DFT

$$X_R(K) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\frac{2\pi Kn}{N} + x_I \sin\frac{2\pi Kn}{N}\right]$$

$$X_I(K) = \sum_{n=0}^{N-1}\left[x_R(n)\sin\frac{2\pi Kn}{N} - x_I \cos\frac{2\pi Kn}{N}\right]$$

IDFT

$$x_R(n) = \frac{1}{N}\sum_{K=0}^{N-1}\left[X_R(K)\cos\frac{2\pi Kn}{N} - X_I(K)\sin\frac{2\pi Kn}{N}\right]$$

$$x_I(n) = \frac{1}{N}\sum_{K=0}^{N-1}\left[X_R(K)\sin\frac{2\pi Kn}{N} + X_I(K)\sin\frac{2\pi Kn}{N}\right]$$

Real and even sequences:

If $x(n)$ is real and even, $x(n) = x(N-n)$, $0 \leq n \leq N-1$

$X_I(K) = 0$, hence the DFT reduce to

$$X(K) = \sum_{n=0}^{N-1} x(n)\cos\frac{2\pi Kn}{N}, \quad 0 \leq K \leq N-1$$

IDFT

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\cos\frac{2\pi Kn}{N}, \quad 0 \leq n \leq N-1$$

Real and Odd Sequences :-

If $x(n)$ is real and odd, $\Rightarrow x(n) = -x(N-n)$, $0 \le n \le N$

$$X_R(K) = 0$$

* DFT

$$X(K) = -j \sum_{n=0}^{N-1} x(n) \sin \frac{2\pi Kn}{N}, \quad 0 \le n \le N-1$$

IDFT

$$x(n) = j \frac{1}{N} \sum_{K=0}^{N-1} X(K) \sin \frac{2\pi Kn}{N}, \quad 0 \le n \le N-1$$

Additional DFT Properties

1.) Time Reversal of a sequence —

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

* then $x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-K))_N = X(N-K)$

2.) Circular time shift of a sequence —

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $x((n-l))_N \xleftrightarrow[N]{DFT} X(K) e^{\frac{-j2\pi Kl}{N}}$

3.) Circular Frequency Shift —

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then $x(n) e^{\frac{j2\pi ln}{N}} \xleftrightarrow[N]{DFT} X((K-l))_N$

4.) Complex-Conjugate Properties —

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

then

$x^*(n) \xleftrightarrow[N]{DFT} X^*((-K))_N = X^*(N-K)$

5.) Circular correlation —

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

$y(n) \xleftrightarrow[N]{DFT} Y(K)$

then $\tilde{r}_{xy}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xy}(K) = X(K) Y^*(K)$

where $\tilde{r}_{xy}$ is the circular correlation sequence, defined as $\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n) y^*((n-l))_N$

6.) Multiplication of two sequences

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$

then $x_1(n) \cdot x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N} X_1(K) \circledN X_2(K)$

7.) Parseval's Theorem

If $x(n) \xleftrightarrow[N]{DFT} X(K)$

$y(n) \xleftrightarrow[N]{DFT} Y(K)$

then $\sum_{n=0}^{N-1} x(n) y^*(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) Y^*(K)$

Multiplication of two DFTs and Circular Convolution

We have two finite-duration sequences of length N, $x_1(n)$ and $x_2(n)$.

The N-point DFT is

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n)\, e^{-j\frac{2\pi kn}{N}}, \quad K = 0, 1, 2, \ldots, N-1$$

$$X_2(K) = \sum_{n=0}^{N-1} x_2(n)\, e^{-j\frac{2\pi Kn}{N}}, \quad K = 0, 1, 2, \ldots, N-1$$

We have

$$X_3(K) = X_1(K) \cdot X_2(K), \quad K = 0, 1, 2, \ldots, N-1$$

IDFT

$$x_3(m) = \frac{1}{N} \sum_{K=0}^{N-1} X_3(K) \cdot e^{j\frac{2\pi Km}{N}},$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X_1(K) \cdot X_2(K) \cdot e^{j\frac{2\pi Km}{N}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{K=0}^{N-1} e^{j2\pi K(m-n-l)/N} \right]$$

We finally obtain the desired expression as:

$$\boxed{x_3(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2(m-n)_N, \quad m = 0, 1, \ldots N-1}$$

Q.) Perform the circular convolution of the following two sequence:

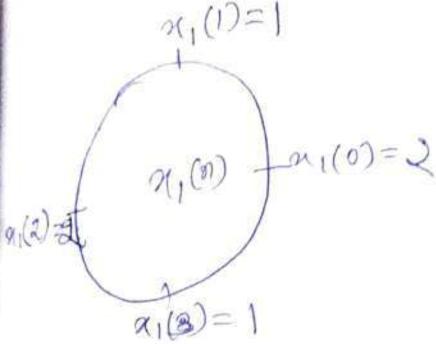
$$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

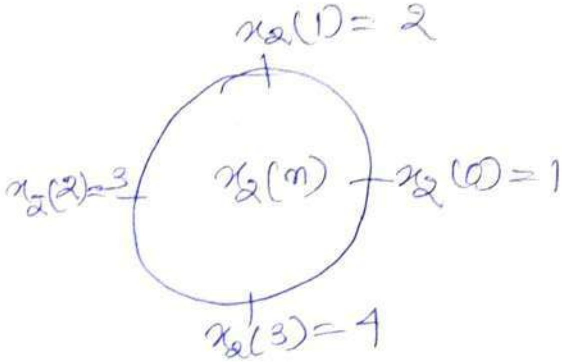
$$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

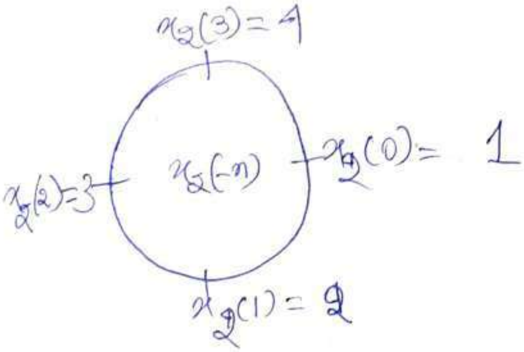
Ans- $x_3(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2(m-n)_N$

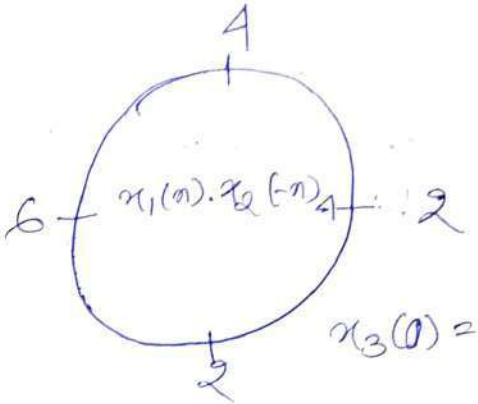
Beginning with $m=0$, we have

$$x_3(0) = \sum_{n=0}^{3} x_1(n) \cdot x_2(-n)_N$$

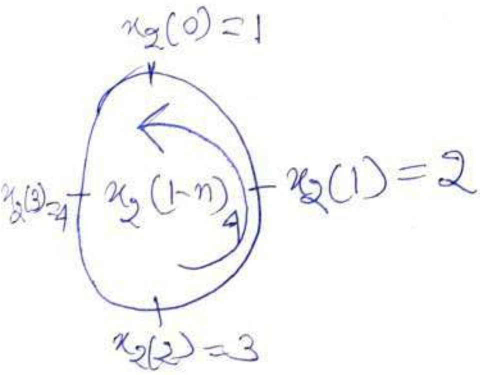


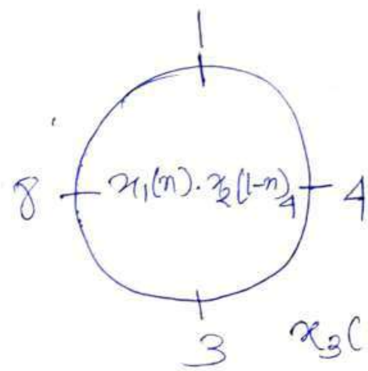




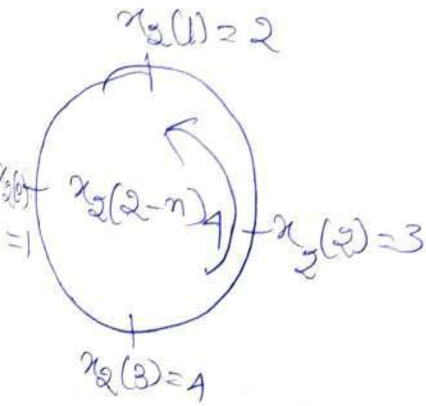


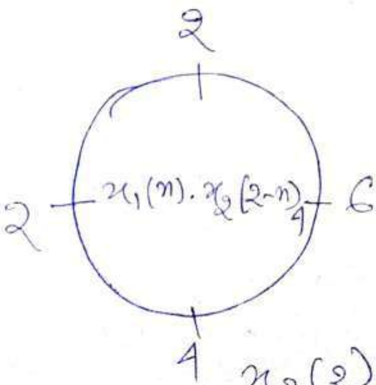
$x_3(0) = 14$



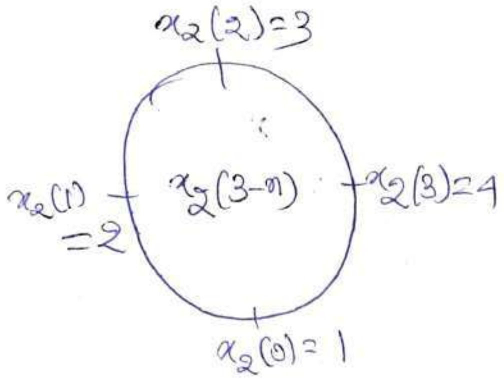


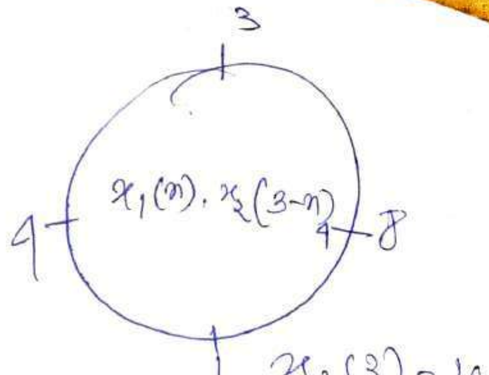
$x_3(1)=16$





$x_3(2)=14$





$x_3(3)=16$

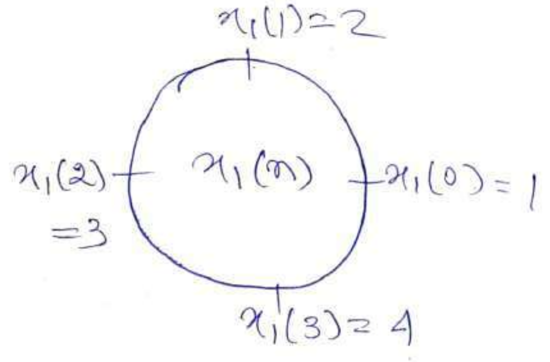
$\therefore x_3(n) = \{\underset{\uparrow}{14}, 16, 14, 16\}$

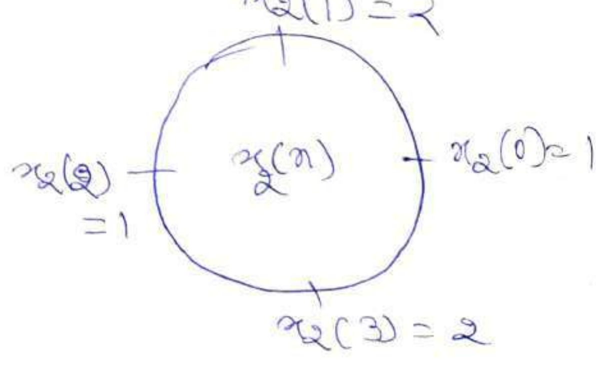
Q.) Find the circular convolution of the following two sequences:

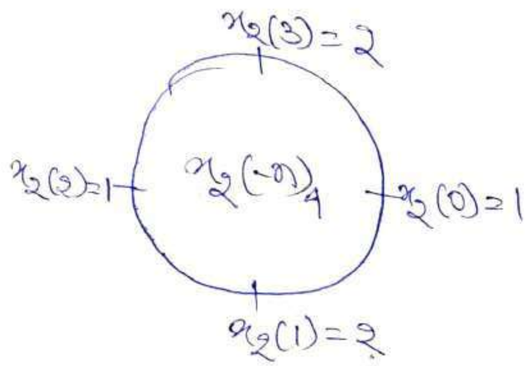
$x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$ and $x_2(n) = \{\underset{\uparrow}{1}, 2, 1, 2\}$

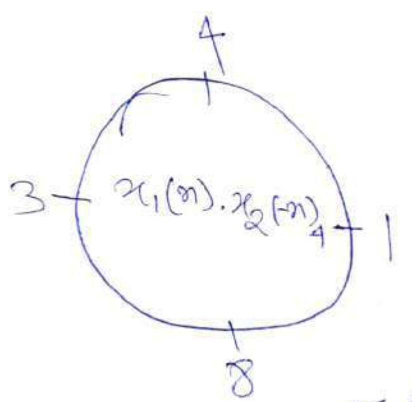
Ans- $x_3(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2(m-n)_N$

$x_3(0) = \sum_{n=0}^{3} x_1(n) \cdot x_2(-n)_4$

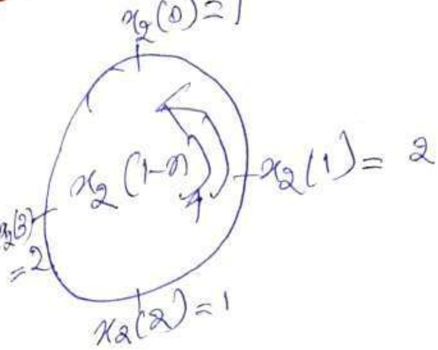


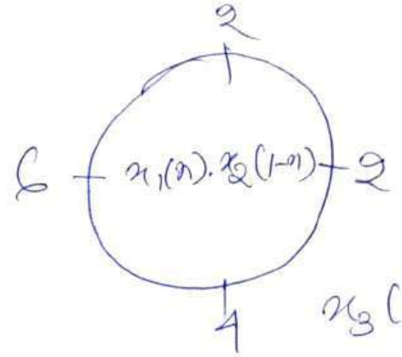




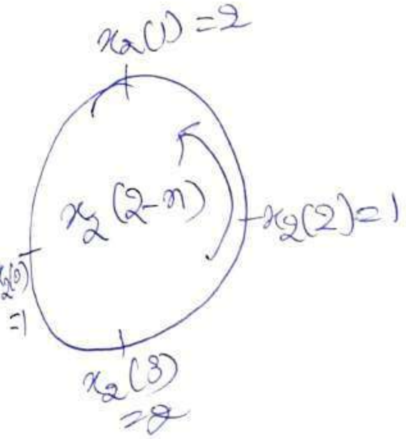


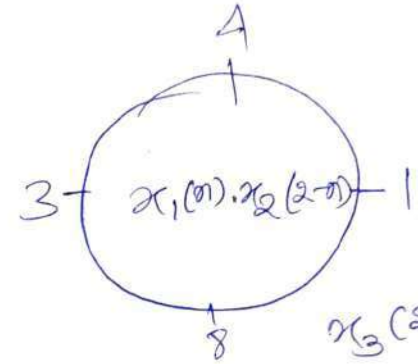
$x_3(0)=16$



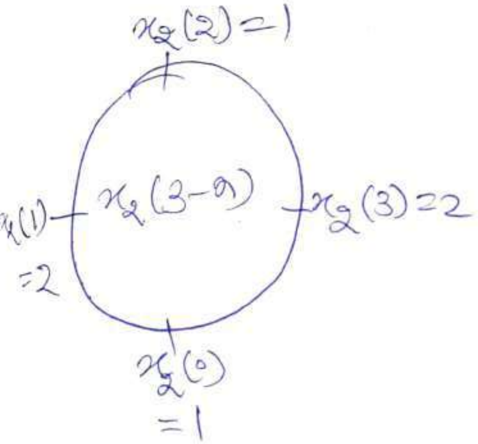


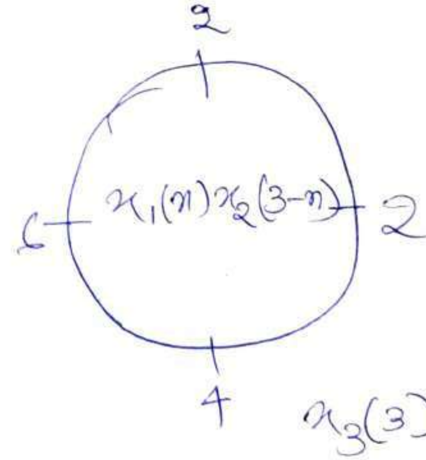
$x_3(1)=14$





$x_3(2)=16$





$x_3(3)=14$

$\therefore\ x_3(n)=\{\underset{\uparrow}{16},14,16,14\}$

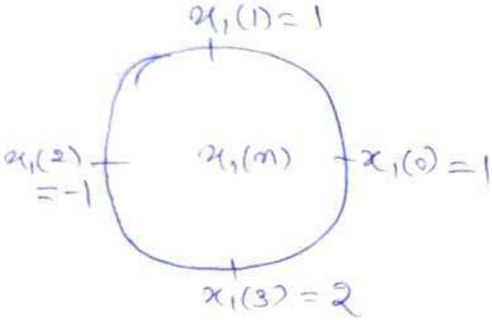
Q) Find the circular convolution of two finite duration sequences:

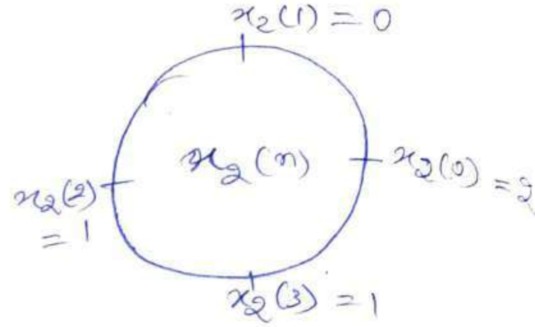
$x_1(n)=\{1,1,-1,2\}$

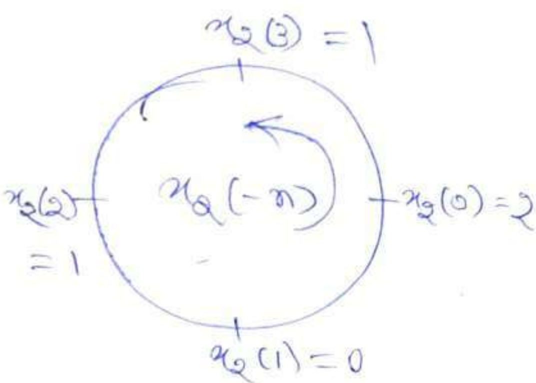
$x_2(n)=\{2,0,1,1\}$

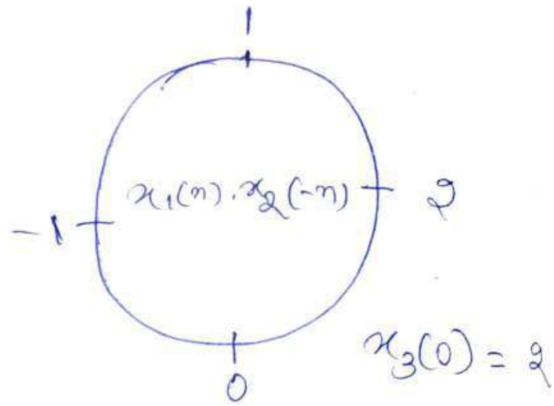
Ans $x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n)$.

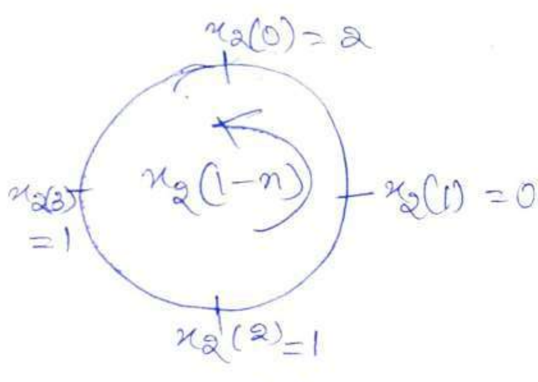
For $m = 0$, $x_3(0) = \sum_{n=0}^{3} x_1(n)\, x_2(-n)$.

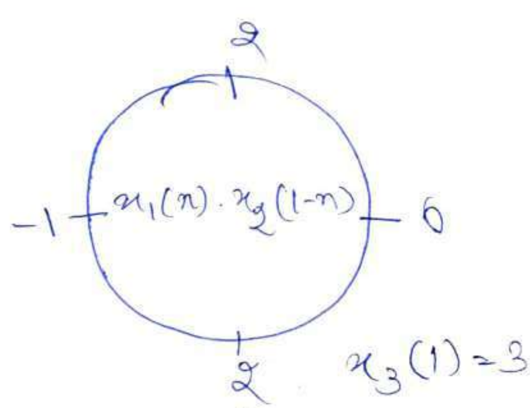


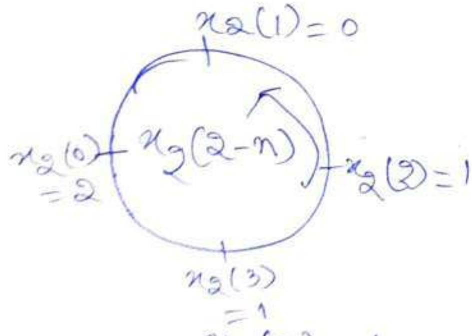


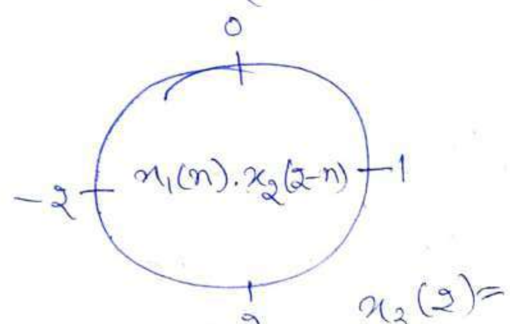


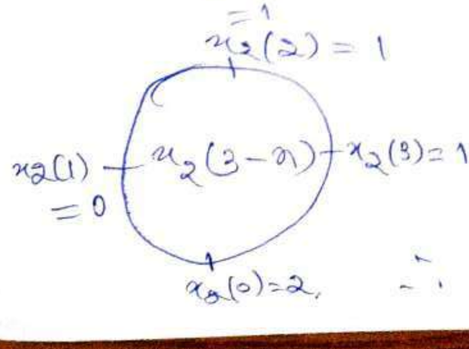


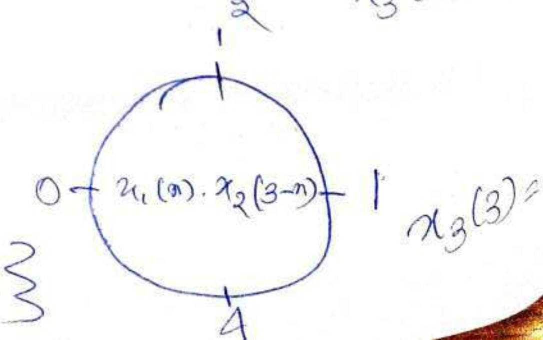












$\therefore x_3(m) = \{ \underset{\uparrow}{2}, 3, 1, 6 \}$

Q.) By means of DFT and IDFT, determine the sequence corresponding circular convolution of $x_1(n) = \{ \underset{\uparrow}{2}, 1, 2, 1 \}$ and $x_2(n) = \{ \underset{\uparrow}{1}, 2, 3, 4 \}$

Ans- First use DFT of $x_1(n)$.

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n) \cdot e^{-j\frac{2\pi n K}{N}}, \quad K = 0, 1, \ldots, N-1.$$

$$\Rightarrow X_1(K) = \sum_{n=0}^{3} x_1(n) \cdot e^{-j\frac{2\pi n K}{N}}, \quad K = 0, 1, \ldots, N-1$$

$$X_1(0) = \sum_{n=0}^{3} x_1(n) \cdot e^{0} = x_1(0) + x_1(1) + x_1(2) + x_1(3)$$

$$= 2 + 1 + 2 + 1 = 6$$

$$X_1(1) = \sum_{n=0}^{3} x_1(n)\, e^{-j\frac{2\pi n}{N}} = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi n}{4}}$$

$$= x_1(0) \cdot e^{0} + x_1(1) \cdot e^{-j\frac{2\pi}{4}} + x_1(2) \cdot e^{-j\frac{2\pi \cdot 2}{4}} + x_1(3) e^{-j\frac{2\pi \cdot 3}{4}}$$

$$= 2 + 1 \cdot (\cos \pi/2 - j \sin \pi/2) + 2 (\cos \pi - j \sin \pi)$$
$$+ 1 \cdot \left(\cos \tfrac{3}{2}\pi - j \sin \tfrac{3\pi}{2}\right)$$

$$= 2 + (-j) + 2(-1) - 0 + (0 + j)$$

$$= \cancel{2} - \cancel{j} - \cancel{2} + \cancel{j} = 0$$

$$X_1(2) = \sum_{n=0}^{3} x_1(n) \cdot e^{-j\frac{2\pi 2n}{N}} = \sum_{n=0}^{3} x_1(n) \cdot e^{-j\pi n}$$

$$= x_1(0) + x_1(1) \cdot e^{-j\pi} + x_1(2) \cdot e^{-j2\pi} + x_1(3)\, e^{-j3\pi}$$

$$= 2 + 1 \cdot (\cos\pi - j \sin\pi) + \cancel{x_1(2)} \cdot 2 [\cos 2\pi - j \sin 2\pi]$$
$$+ 1 \cdot (\cos 3\pi - j \sin 3\pi)$$

$$X_1(3) = \sum_{n=0}^{3} x_1(n).\, e^{-j\frac{2\pi 3n}{N}} = \sum_{n=0} x_1(n)\, e^{-j\frac{2\pi 3n}{4}}$$

$$= \sum_{n=0}^{3} x_1(n).\, e^{-j\frac{3\pi n}{2}}$$

$$= x_1(0) + x_1(1).\, e^{-j\frac{3\pi}{2}} + x_1(2).\, e^{-j3\pi} + x_1(3) e^{-j\frac{9\pi}{2}}$$

$$= 2 + 1.\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right) + 2.\,(\cos 3\pi - j\sin 3\pi)$$

$$+ 1.\left(\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right)$$

$$= 2 + 1(0 - (-j)) + 2.\,(-1 - 0) + 1(0 - 1j)$$

$$= \cancel{2} + \cancel{1j} - \cancel{2} - \cancel{1j} = 0.$$

DFT of $x_2(n)$

$$X_2(n) = \sum_{n=0}^{3} x_2(n).\, e^{-j\frac{2\pi nK}{N}}$$

$$= x_2(0)\, e^{0} + x_2(1) e^{-j\frac{\pi K}{2}} + x_2(2)\, e^{-j\pi K} + x_2(3) e^{-j\frac{3\pi K}{2}}$$

$$= 1 + 2e^{-j\frac{\pi K}{2}} + 3e^{-j\pi K} + 4e^{-j\frac{3\pi K}{2}}$$

$$X_2(0) = 1 + 2 + 3 + 4 = 10$$

$$X_2(1) = 1 + 2e^{-j\pi/2} + 3.\, e^{-j\pi} + 4e^{-j3\pi/2} = -2 + j2$$

$$X_2(2) = 1 + 2(\cos\pi/2 - j\sin\pi/2) + 3(\cos\pi - j\sin\pi) + 4\left(\cos\frac{3\pi}{2} \ldots\right)$$
$$= -2$$

$$X_2(3) = -2 - j2$$

When we multiply the two DFTs, we obtain the product:

$X_3(K) = X_1(K) \cdot X_2(K)$

$\Rightarrow X_3(0) = 60, \quad X_3(1) = 0, \quad X_3(2) = -4, \quad X_3(3) = 0.$

IDFT

$x_3(n) = \frac{1}{N} \sum_{K=0}^{N-1} X_3(K) \cdot e^{j\frac{2\pi n K}{N}}, \quad n = 0, 1, 2, \ldots, N-1$

$= \frac{1}{4} \sum_{K=0}^{3} X_3(K) \cdot e^{j\frac{2\pi n K}{4}}$

$= \frac{1}{4}(60 - 4e^{j\pi n})$

$x_3(0) = 14, \quad x_3(1) = 16, \quad x_3(2) = 14, \quad x_3(3) = 16.$

Circular convolution

$x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$

$x_1(n) \, (N) \, x_2(n) \xleftrightarrow[N]{DFT} X_1(K) \cdot X_2(K).$

Q.) Determine the sequence $y(n)$ that result use of 4-point DFT of sequence $h(n) = \{1, 2, 3\}$
$x(n) = \{1, 2, 2, 1\}$

Ans- $h(n) \xleftrightarrow{DFT} H(K)$

$x(n) \xleftrightarrow{DFT} X(K)$

$y(n) \xleftrightarrow{DFT} H(K) \cdot X(K).$

$H(K) = W_4(K) \cdot h_4(K)$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+3+0 \\ 1-2j-3+0 \\ 1-2j+3-0 \\ 1+2j-3+0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2-2j \\ 0 \\ -2+2j \end{bmatrix}$$

$$X_4(K) = W_4(K) \cdot x_4(n)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$X_4(K) = \begin{bmatrix} 6 \\ 1-2j-2+j \\ 1-2+2-1 \\ 1+2j-2-j \end{bmatrix} = \begin{bmatrix} 6 \\ -1-j \\ 0 \\ -1+j \end{bmatrix}$$

$$Y(K) = X(K) \cdot H(K)$$

$Y(0) = 36$, $Y(1) = 4j$, $Y(2) = 0$, $Y(3) = -4j$.

Unit-V **Fast Fourier Transform Algorithm and Digital Filters**

→ ~~The~~ **Efficient Computation of the DFT : FFT Algorithm**

→ The importance of DFT in various digital signal processing applications, such as linear filtering, correlation analysis, and spectrum analysis, its efficient computation is a topic that has received considerable attention by many mathematicians, ~~or~~ engineers, and applied scientists.

→ The computational problem for the DFT is to compute the sequence $\{X(K)\}$ of N-complex-valued numbers given another sequence of data $\{x(n)\}$ of length N, according to the formula

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}, \quad 0 \leq K \leq N-1$$

where $W_N = e^{-j2\pi/N}$

IDFT

$$x(n) = \sum_{K=0}^{N-1} X(K) \cdot W_N^{-Kn}, \quad 0 \leq n \leq N-1.$$

* FFT consumes less time and calculation.

* FFT is faster than DFT.

We observed that for each value of K, direct computation of X(K) involves N-complex multiplication (4N real multiplication) and N-1 complex addition (4N-2 real additions). Consequently, to compute all N values of the

DFT requires N^2 complex multiplication and $N^2 - N$ complex additions

→ Direct computation of DFT is basically inefficient, primarily because it does not exploit the symmetry and periodicity properties of the phase factor W_N.

- Symmetry property: $W_N^{K+N/2} = -W_N^K$
- Periodicity property: $W_N^{K+N} = W_N^K$

→ The computationally efficient algorithm known as Fast Fourier Transform (FFT) algorithm, exploit these two basic properties of the phase factor

→ Direct Computation of the DFT

For a complex-valued sequence $x(n)$ of N-points, the DFT may be expressed as:

$$X_R(K) = \sum_{n=0}^{N-1} \left[x_R(n) \cos \frac{2\pi K n}{N} + x_I(n) \sin \frac{2\pi K n}{N} \right]$$

$$X_I(K) = \sum_{n=0}^{N-1} \left[x_R(n) \sin \frac{2\pi K n}{N} - x_I(n) \cos \frac{2\pi K n}{N} \right]$$

- The direct computation of N-point DFT requires:

1. $2N^2$ evaluations of trigonometric functions
2. $4N^2$ real multiplications
3. $4N(N-1)$ real additions
4. A number of indexing and addressing operations

Divide and Conquer Approach to computation of DFT

The development of computationally efficient algorithms for the DFT is made possible if we adopt a divide-and-conquer approach. This approach is based on the decomposition of an N-point DFT into successively smaller DFTs. This basic approach leads to a family of computationally efficient algorithms known as FFT algorithms.

Radix-2 FFT Algorithm

In the previous section, we described algorithms for efficient computation of the DFT based on the divide-and-conquer approach. Such an approach is applicable when the number N of data points is not a prime. The approach is very efficient when N is highly composite, i.e., when N can be factored as $N = r_1 r_2 r_3 \cdots\cdots r_v$ where the $[r_j]$ are prime!

In cases where $r_1 = r_2 = \cdots\cdots = r_v \equiv r$, so that $N = r^v$, in such case the DFTs are of size r, so the computation of N-point DFT has a regular pattern. The number 'r' is called the radix of the FFT algorithm.

Let $N = 2^v$ point DFT by divide-and-conquer approach. We select $M = N/2$ and $L = 2$. This selection results in a split of N-point data sequence $f_1(n)$ and $f_2(n)$, corresponding to the even-numbered and odd-numbered sample of $x(n)$.

$$f_1(n) = x(2n)$$
$$f_2(n) = x(2n+1),\ n = 0, 1, \ldots, \frac{N}{2}-1$$

Now, N-point DFT

$$X(K) = \sum_{n=0}^{N-1} x(n).W_N^{Kn},\ K = 0, 1, \ldots, N-1$$

$$= \sum_{n=\text{even}} x(n) W_N^{Kn} + \sum_{n=\text{odd}} x(n) W_N^{Kn}$$

$$= \sum_{m=0}^{N/2-1} x(2m).W_N^{2mK} + \sum_{m=0}^{N/2-1} x(2m+1) W_N^{K(2m+1)}$$

We ~~substitute~~ substitute $W_N^2 = W_{N/2}$

$$= \sum_{m=0}^{N/2-1} f_1(m) W_N^{2mK} + \sum_{m=0}^{N/2-1} f_2(m) W_{N/2}^{Km}$$

$$= F_1(K) + W_N^K F_2(K),\ K = 0, 1, \ldots, N-1$$

$$X(K) = F_1(K) + W_N^K F_2(K),\ K = 0, 1, \ldots, N-1$$

$$X\left(K+\frac{N}{2}\right) = F_1(K) + W_N^K F_2(K),\ K = 0, 1, 2, \ldots N-1$$

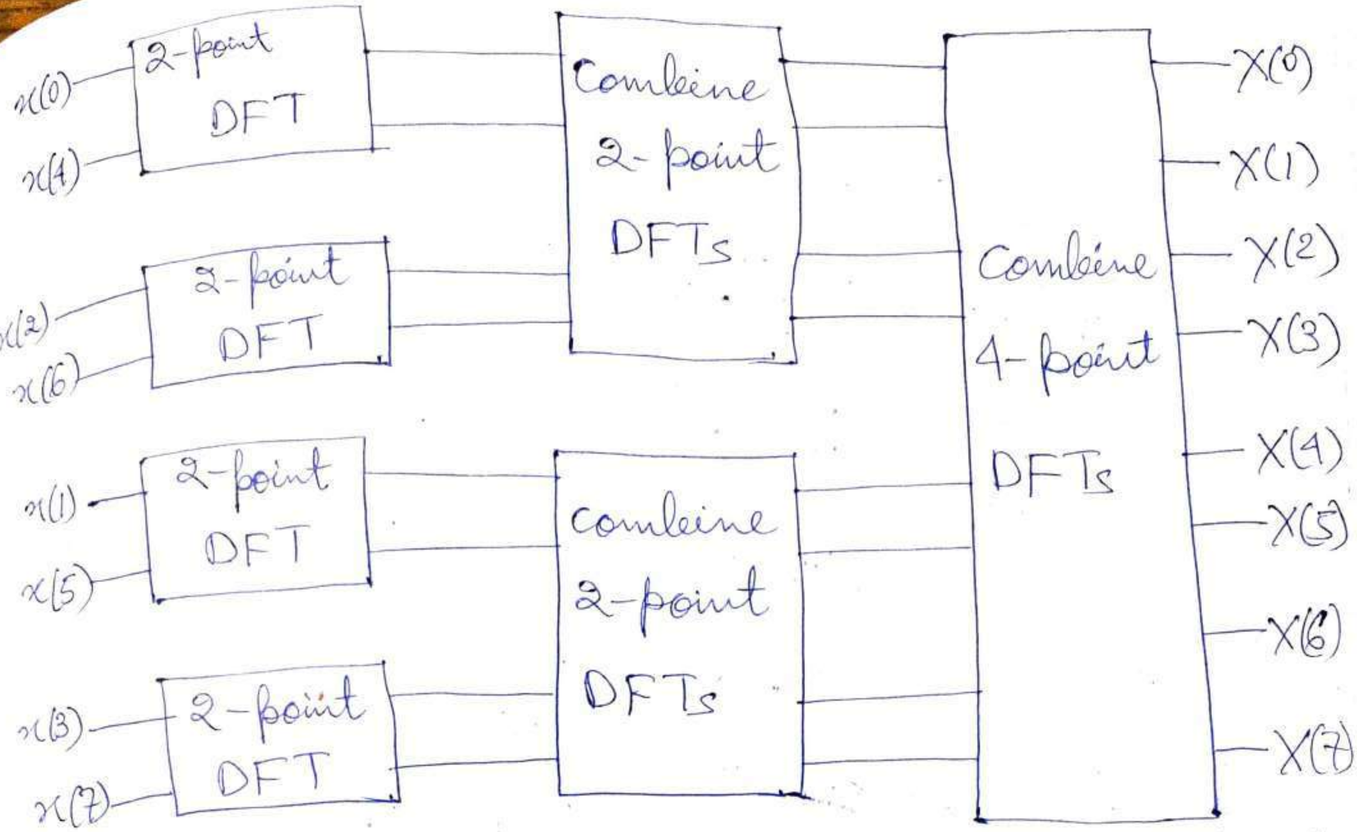
If $G_1(K) = F_1(K),\ K = 0, 1, \ldots \frac{N}{2}-1$

$G_2(K) = W_N^K F_2(K),\ K = 0, 1, \ldots, \frac{N}{2}-1$

Then the DFT $X(K)$ may be expressed as

$$X(K) = G_1(K) + G_2(K),\ K = 0, 1, \ldots, \frac{N}{2}-1$$

$$X\left(K+\frac{N}{2}\right) = G_1 K - G_2(K),\ K = 0, 1, \ldots, \frac{N}{2}-1$$

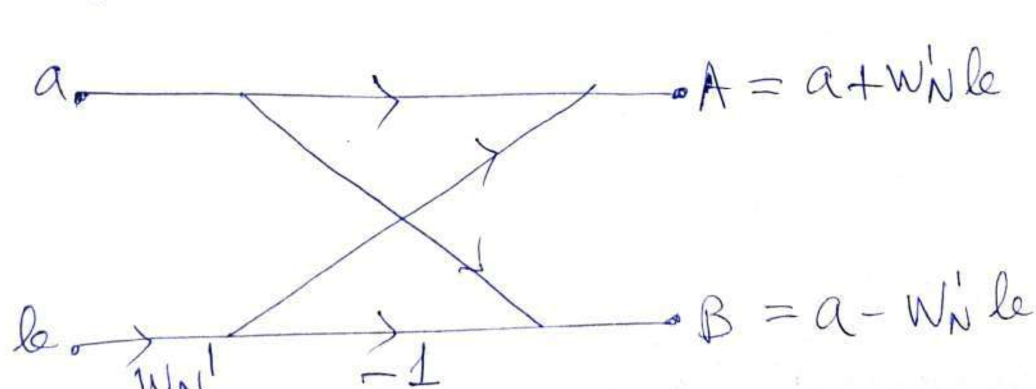


(Three stages in the computation of an $N = 8$-point DFT)

Decimation-in-Time FFT (DIT-FFT)

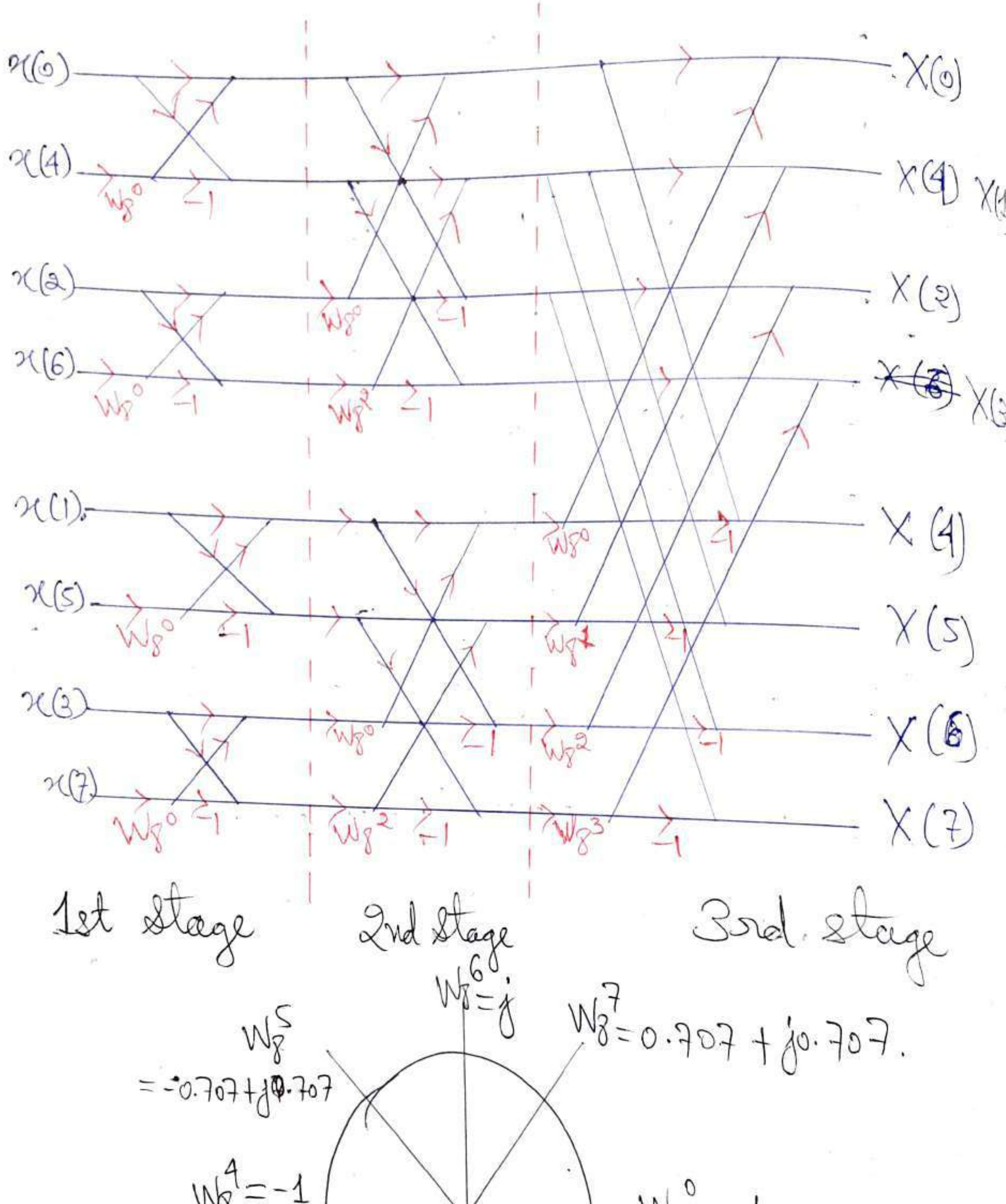
* In general, each butterfly involves one complex ~~addition~~ multiplication and two complex additions.

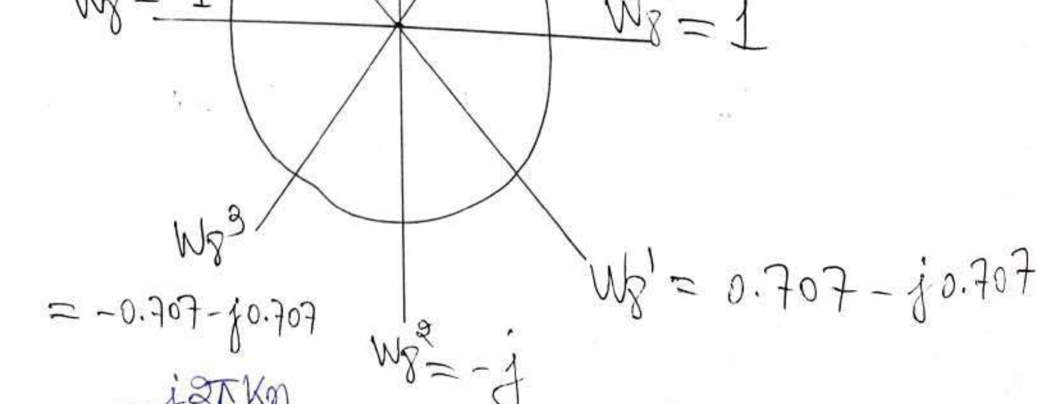
→ Basically we use butterfly computation in the DIT-FFT algorithm.



Given sequence $x(n)$, then we get $X(K)$.

8-point DIT-FFT Algorithm





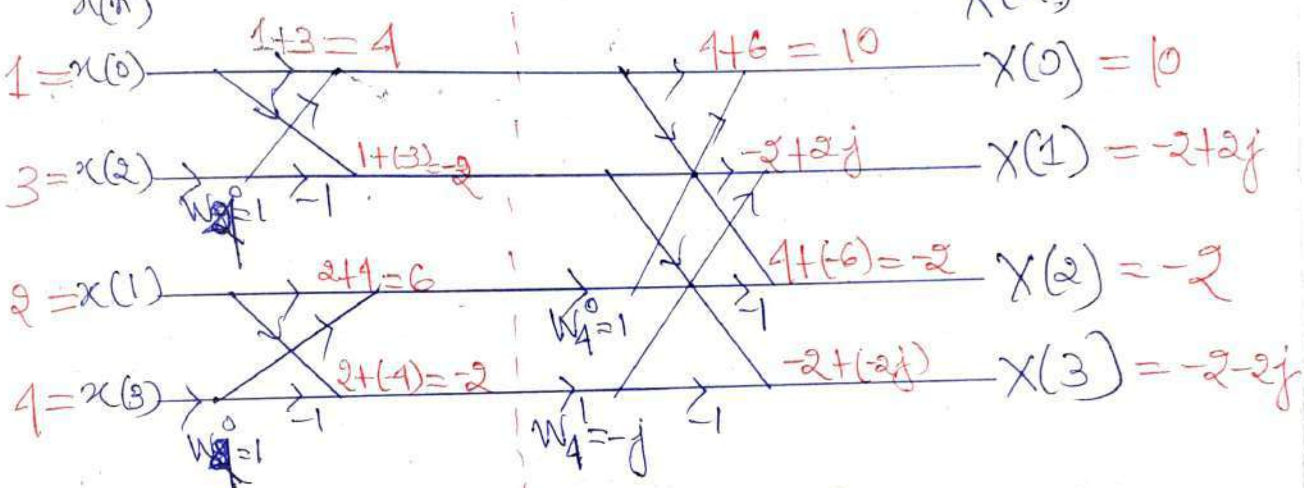
$$W_N^{Kn} = e^{-j\frac{2\pi Kn}{N}}$$

Q.) Determine the DFT of the sequence using Radix-2 DIT-FFT algorithm:

$$x(n) = \{1, 2, 3, 4\}$$

Ans- $N = 4 = 2^2$

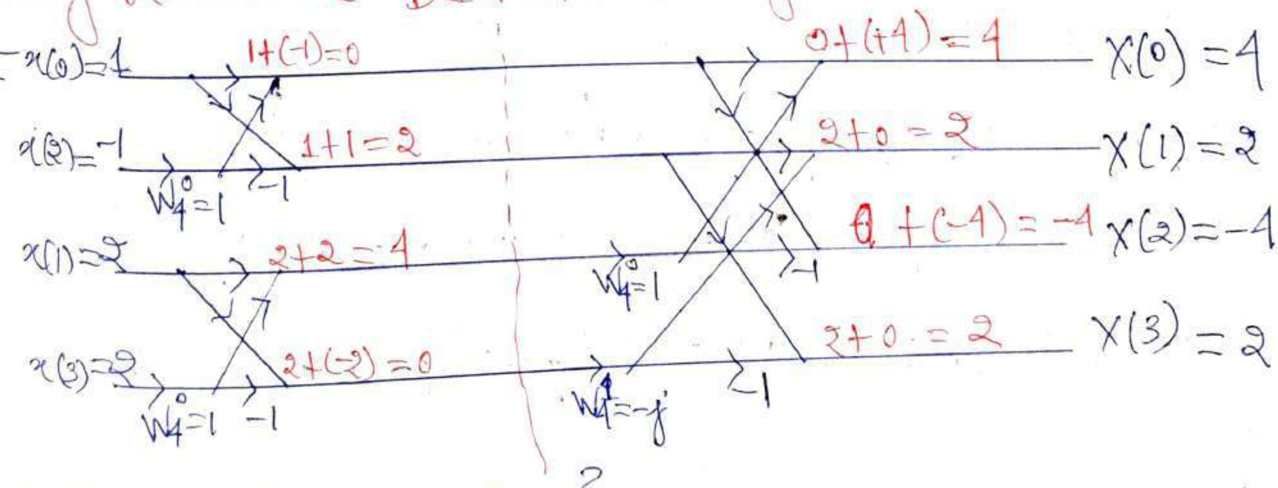
	B.R	
$0 \to 00$	00	$\to 0$
$1 \to 01$	10	$\to 2$
$2 \to 10$	01	$\to 1$
$3 \to 11$	11	$\to 3$



$\therefore X(K) = \{10, -2+2j, -2, -2-2j\}$

Q.) Determine the DFT of the sequence: $x(n) = \{1, 2, -1, 2\}$ using Radix-2 DIT-FFT algorithm.

Ans-



$\therefore X(K) = \{4, 2, -4, 2\}$

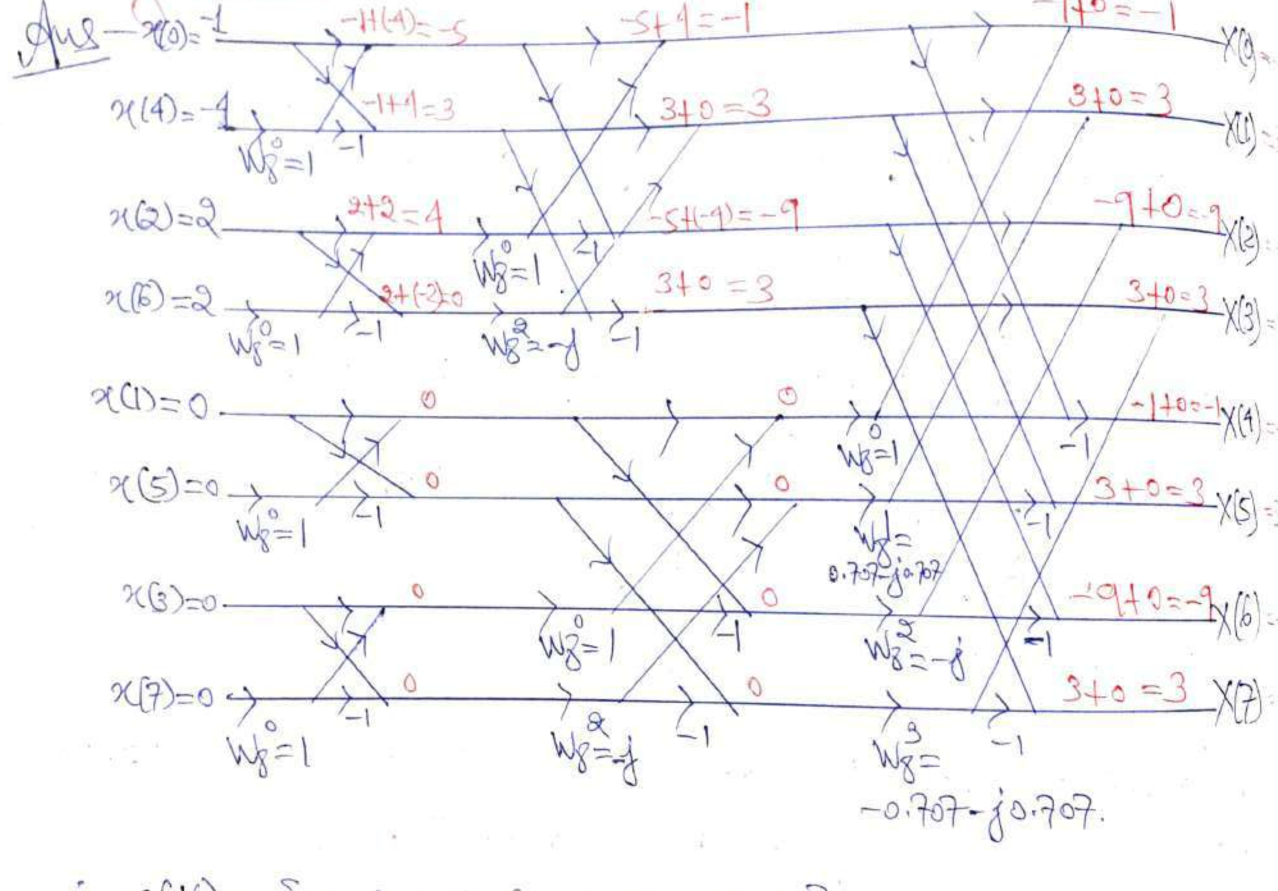
Q.) Using Radix-2 DIT-FFT algorithm find the DFT of the following sequences:

i) $x(n) = \{0, 1, 2, 3\}$.

ii) $x(n) = \{3, 2, 1, 2\}$

Q.) Find the DFT of the 8-point sequence given by $x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$ using Radix-2 DIT-FFT algorithm.

Ans-

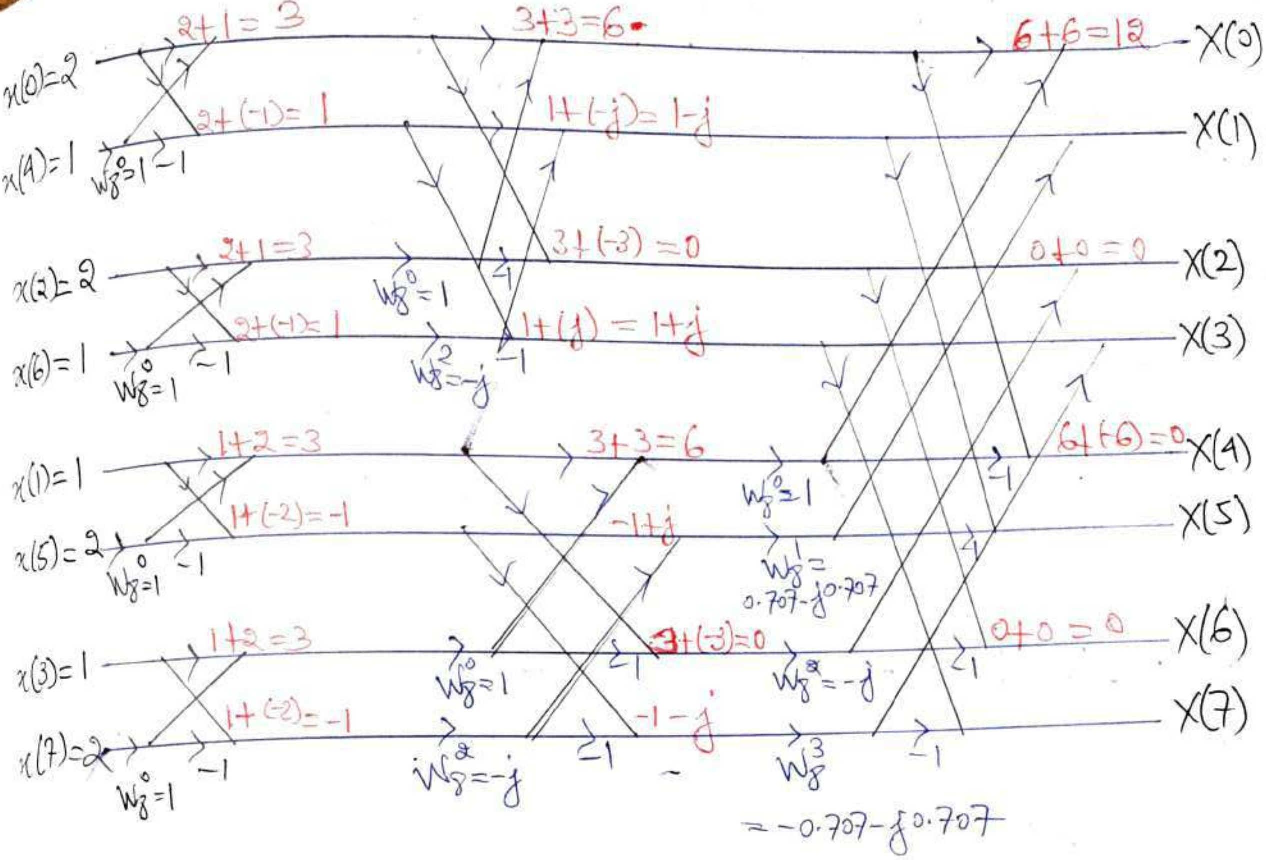


$\therefore X(K) = \{-1, 3, -9, 3, -1, 3, -9, 3\}$

Q.) Find the 8-point DFT of the sequence which is given by: $x(n) = \{2, 1, 2, 1, 1, 2, 1, 2\}$. by using Radix-2 DIT-FFT algorithm.

Ans- $x(n) = \{2, 1, 2, 1, 1, 2, 1, 2\}$

$x(0)$ 1 2 3 4 5 6 7



$X(0) = 12$

$X(1) = (1-j) + [(-1+j)(0.707-j0.707)]$

$= (1-j) + [-0.707 + j0.707 + 0.707j - j^2 0.707]$

$= (1-j) + [-j^2 0.707 + 2j0.707 - 0.707]$

$X(2) = 0$

$X(3) = (1+j) + [(-1-j)(-0.707-j0.707)]$

$= (1+j) + [0.707 + j0.707 + j0.707 + j^2 0.707]$

$= (1+j) + [j^2 0.707 + 2j0.707 + 0.707]$

$X(4) = 0$

$X(5) = ((1-j) + [(-1+j)(0.707-j0.707)]) \times -1$

$= \{(1-j) + (-j^2 0.707 + 2j0.707 - 0.707)\} \times -1$

$= (1-j) - (-j^2 0.707 + 2j0.707 - 0.707)$

$X(6) = 0$

$X(7) = \{(1+j) + (j^2 0.707 + 2j0.707 + 0.707)\} \times -1$

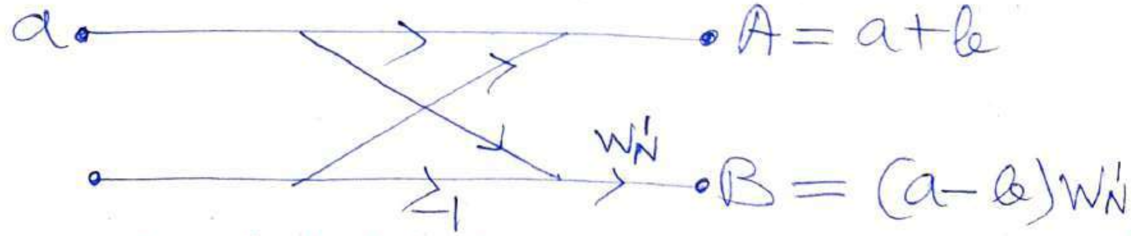
$= (1+j) - (j^2 0.707 + 2j0.707 + 0.707)$

Q.) Find the 8-point DFT of the given sequence $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$ using Radix-2 DIT-FFT algorithm.

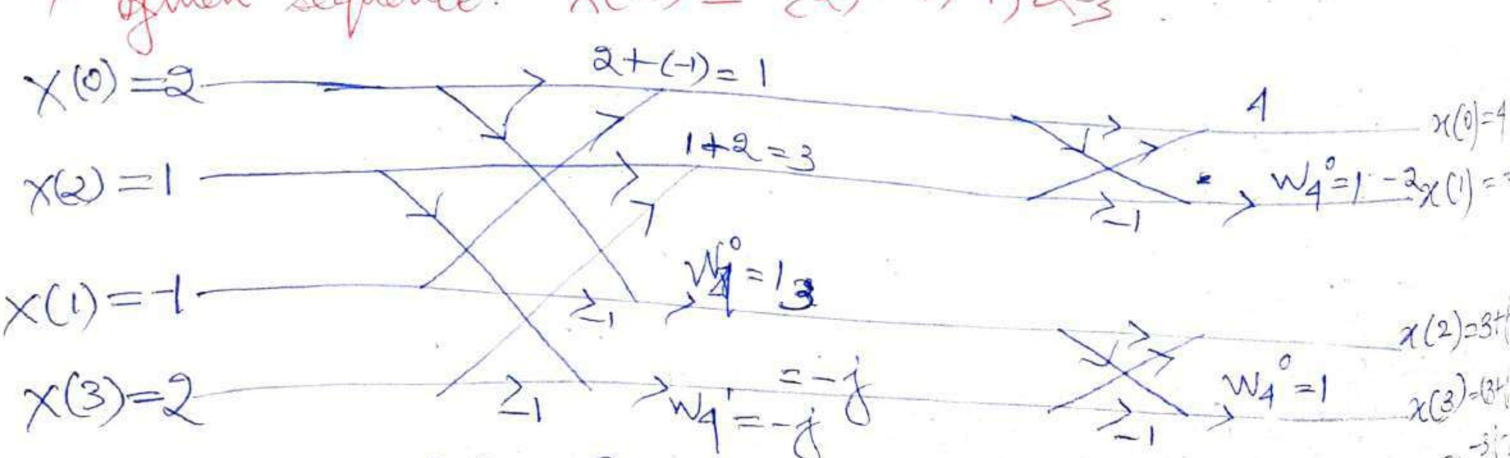
* **Zero padding** :- The method of extending signals by adding zeros is known as zero padding.
  → Zero padding enables you to obtain more accurate amplitude estimates of resolvable signal components.

* **Twiddle Factor** :- It is defined as any of the trigonometric constant coefficients that are multiplied by the data in the course of the algorithm.
  It is represented as : $W_N^{kn} = e^{-j\frac{2\pi kn}{N}}$

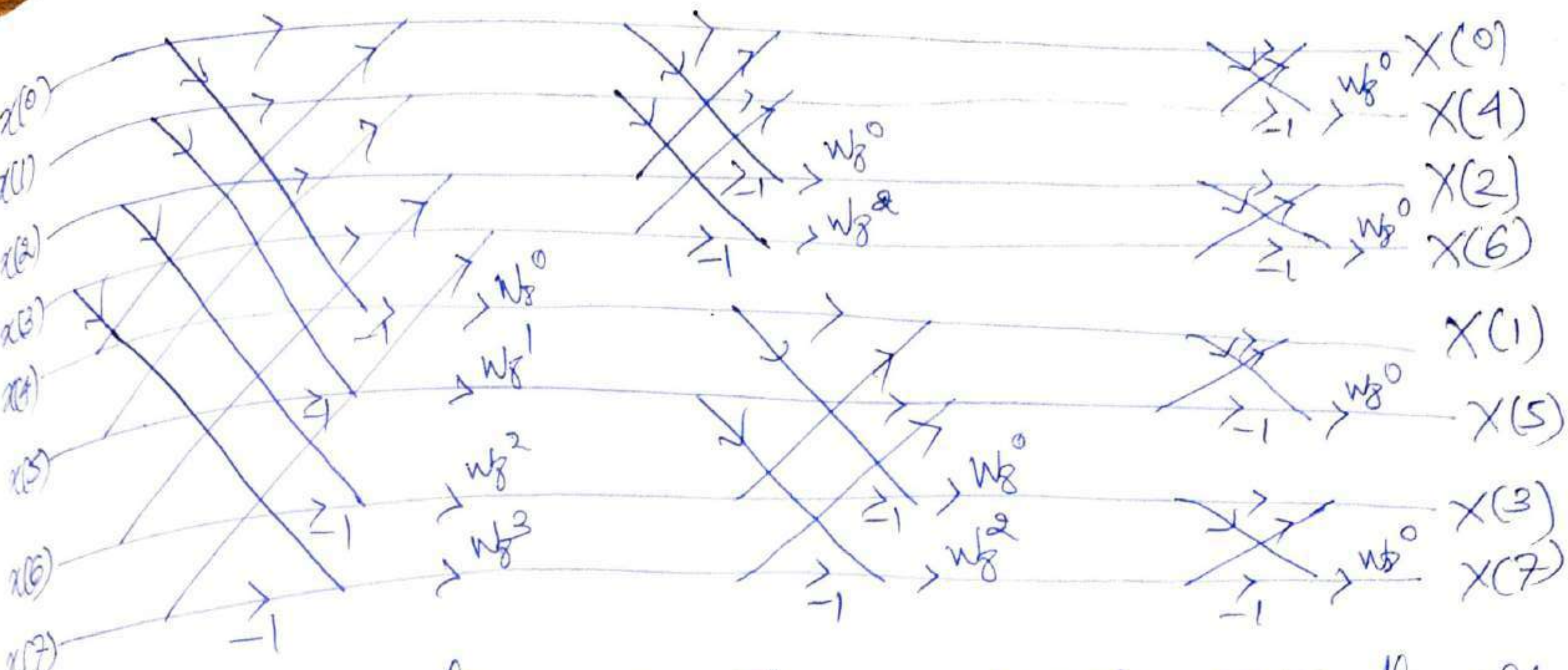
Decimation In Frequency (DIF)-FFT Radix-2 Algorithm



Q.) Calculate the 4-point DFT using DIF-FFT of the given sequence: $X(K) = \{2, -1, 1, 2\}$



$\therefore x(n) = \{$



N=8 Decimation - In - Frequency (DIF) - FFT Algorithm

Applications of FFT Algorithms

1) Effecient computation of the DFT of two Real sequences.

2) Effecient computation of the DFT of a 2N-point Real Sequence

3) Use of FFT Algorithm in linear filtering and correlation.

Total Pages—4

VI—Sem/E & TC/AE & I/ 2017(S)(New)

DIGITAL SIGNAL PROCESSING

(Code : ETT-603/AIT-603)

Full Marks : 70

Time : 3 hours

Answer any five questions

Figures in the right-hand margin indicate marks

1. (*a*) What is the difference between Deterministic and Random signal ? 2

   (*b*) What is signal processing. Draw the block diagram and explain digital signal processing system. 5

   (*c*) $x(n) = e^{2n}u(n)$. Determine the signal is energy signal or power signal. 7

2. (*a*) Determine $x(n) = u(n+1)$ is a casual signal or non-casual signal. 2

(*Turn Over*)

(*b*) If

$$x(n) = \begin{cases} 1 \text{ for } n = -1, 0, 2, 3 \\ -1 \text{ for } n = -2, 1 \\ 0 \text{ otherwise} \end{cases}$$

then find out $x(n+2)$, $x(-n)$, $x(-n+3)$, $x(-n-1)$. 5

(*c*) Define linear and non-linear system and prove that

$$y(n) = 2x(n) + \frac{1}{x(n-1)}$$

is a linear system or non-linear system. 7

3. (*a*) What are the necessary condition for stable system ? 2

(*b*) Draw and explain principle of analog to digital converter. 5

(*c*) Find out the convolution between two signal

$$x(n) = \begin{cases} 1 & \text{for } n = -2, 0, 1 \\ 2 & \text{for } n = -1 \\ 0 & \text{elsewhere} \end{cases}$$

and $h(n) = \delta(n) - \delta(n-1) - \delta(n-2) + \delta(n+1)$. 7

4. (*a*) Write down the properties of convolution. 2

(*b*) Find the *z*-transform and ROC of a system

$$x(n) = a^n u(n) - b^n u(n).$$ 5

(*c*) Write down all the properties of *z*-transform with proof. 7

5. (*a*) Write down all methods are use for find out inverse *z*-Transform. 2

(*b*) Determine the inverse *z*-transform of

$$X(z) = \frac{1}{1 - 1 \cdot 5z^{-1} + 0 \cdot 5z^{-2}}$$

where (*i*) ROC : $|z| > 1$

(*ii*) ROC : $|z| < 0 \cdot 5$

(*iii*) ROC : $0 \cdot 5 < |z| < 1$ 5

(*c*) Find out the forced response of the system described by the equation :

$$y(n) = 0{\cdot}6\, y(n-1) - 0{\cdot}08\, y(n-2) + x(n).$$ 7

6. (*a*) Define DFT. 2

(*b*) Explain the relation of DFT to the other transform. 5

(*c*) Find out the 4-point DFT of

$$x(n) = (-1)^n.$$ 7

7. (*a*) Write down the periodicity and time reversal properties of DFT. 2

(*b*) Derive an expression for DFT in Radin-2 DIF-FFT and justify in case of DIT-FFT the total sequence is contain equal no. of even part and odd part. 5

(*c*) Find the 8-point DFT of the sequence given by $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$ using Radin-2 DIT FFT algorithm. 7

VI/SEM/E&TC/AE&I/2016
(S) New

DIGITAL SIGNAL PROCESSING

Sub Code-**ETT-603**

Full Marks : 70

Time : 3 hours

Answer any **five** questions

The figures in the right-hand margin indicate marks

1. (*a*) Define Time variant and Time invariant system. 2

(*b*) Explain the different properties of z transform. 5

(*c*) Determine the causal signal $x(n)$ having the z transform $x(z)=\frac{1}{(1-2z^{-1})(1-z^{-1})^2}$ using partial fraction expansion method. 7

2. (*a*) What is Twiddle Factor ? 2

(*b*) Find $H(z)$, the system function for the following : 5

(*i*) $y(n)-3y(n-1)+2y(n-2)=x(n)-x(n-1)$

(*Turn Over*)

(ii) $y(n) = x(n) + x(n-1) - 2x(n-2) + 3x(n-3)$

(c) Find the z transform and ROC of the sequence

$$x(n) = 2^n . \sin\left(\frac{n\pi}{4}\right) . u(n)$$ 7

3. (a) Define Zero padding. 2

(b) Compute the convolution $y(n)$ and correlation $r(n)$ for the given signals : 5

$$x_1(n) = \{1, 2, 3, 4\}$$
$$\qquad\uparrow$$

$$x_2(n) = \{1, 2, 3, 4\}$$
$$\qquad\qquad\uparrow$$

(c) Find the circular convolution of two finite duration sequences 7

$$x_1(n) = \{1, 1, -1, 2\}$$
$$x_2(n) = \{2, 0, 1, 1\}$$

4. (a) What is the need of signal processing and give any two applications. 2

(*b*) Find the step response of the following differential equation : 5

$$y(n)-5y(n-1)+6y(n-2)=x(n)$$

(*c*) Compute the inverse *z* transform of

$$x(z)=\frac{z\left(1-e^{-T}\right)}{(z-1)\left(z-e^{-T}\right)}$$ 7

5. (*a*) Draw the basic butterfly diagram for DIT-FFT and DIF-FFT. 2

(*b*) Find the *z* transform and ROC of

$$x(n)=(0.4)^{n}u(n)+(0.3)^{n},u(n-4)$$ 5

(*c*) Compute the 4 point DFT of the sequence

$$x(n)=\{0,2,4,6\}$$ 7

6. (*a*) Define Periodic and Aperiodic signals. 2

(*b*) Verify whether the following systems are linear or non-linear : 5

(*i*) $y(nT)=x(nT+T)+x(nT-T)$

(*ii*) $y(n) = x(n+7)$

(*c*) Determine the DFT of the sequence

$$x(n) = \{1, 2, -1, 1\}$$

using DIT-FFT algorithm. 7

7. (*a*) What is Discrete Fourier transform ? 2

(*b*) Determine the IDFT of $X(K) = \{1, 0, 1, 0\}$ 5

(*c*) Using property find the z transform and ROC of

$$x(n) = n \,.\, u(n-1)$$

where, $x(n)$ is causal. 7
